A Diffusion Process Perspective on Posterior Contraction Rates for Parameters

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Abstract

We show that diffusion processes can be exploited to study the posterior contraction rates of parameters in Bayesian models. By treating the posterior distribution as a stationary distribution of a stochastic differential equation (SDE), posterior convergence rates can be established via control of the moments of the corresponding SDE. Our results depend on the structure of the population log-likelihood function, obtained in the limit of an infinite sample sample size, and stochastic perturbation bounds between the population and sample log-likelihood functions. When the population log-likelihood is strongly concave, we establish posterior convergence of a d-dimensional parameter at the optimal rate $(d/n)^{1/2}$. In the weakly concave setting, we show that the convergence rate is determined by the unique solution of a non-linear equation that arises from the interplay between the degree of weak concavity and the stochastic perturbation bounds. We illustrate this general theory by deriving posterior convergence rates for three concrete examples: Bayesian logistic regression models, Bayesian single index models, and over-specified Bayesian mixture models.

1 Introduction

Bayesian inference is one of the central pillars of statistics. In Bayesian analysis, we first endow the parameter space with a prior distribution, which represents a form of prior belief and knowledge; the posterior distribution obtained by Bayes' rule combines this prior information with observations. Fundamental questions that arise in Bayesian inference include consistency of the posterior distribution as the sample size goes to infinity, and from a more refined point of view, the contraction rate of the posterior distribution.

The earliest work on posterior consistency dates back to the seminal work by Doob [8], who demonstrated that, for any given prior, the posterior distribution is consistent for all parameters apart from a set of zero measure. Subsequent work by Freedman [10, 11] provided examples showing that this null set can be problematic for Bayesian consistency in nonparametric problems. In order to address this issue, Schwartz [29] proposed a general framework for establishing posterior consistency for both semiparametric and nonparametric models. Since then, a number of researchers have isolated conditions that are useful for studying posterior disributions [1, 36, 37].

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Moving beyond consistency of the posterior distribution, convergence rates for the posterior density function and associated parameters remain an active area of research. The seminal paper by Ghosal et al. [13] lays out a general framework for analyzing rates of convergence of posterior densities in both finite and infinite dimensional models. Their result relies on a sieve construction, meaning a series of subsets of the full model class such that: (a) the difference between these model subsets and the full model, as measured in terms of mass under the prior, vanishes as the sample size size goes to infinity and (b) the prior distribution puts a sufficient quantity of mass on the neighborhood around the true density function. Drawing on the framework of that paper, various works established the posterior convergence rates of density functions under several statistical models. For instance, the papers [14, 15, 27, 30] provided (adaptive) rates of density function in Dirichlet Process and nonparametric Beta mixture models. Other work [2, 41, 40] established minimax optimal rates for regression function in nonparametric regression models. Related problems include adaptive rates for the density in nonparametric Bayesian inference [7, 12], and posterior contraction rates of density under misspecified models [18]. Other popular general frameworks for analyzing the density functions of posterior distributions include those of Shen and Wasserman [31], and Walker et al. [38].

While our current understanding of convergence of posterior densities is relatively well-understood, there remain various unresolved issues with the corresponding questions for model parameters. On one hand, for certain classes of Bayesian (in)finite mixture models, the precise posterior convergence rates of parameters are well-studied [22, 16]. However, these results rely on the boundedness of the parameter space, which can be problematic due to model misspecification. Another popular example is Latent Dirichlet Allocation model, which is widely applied in machine learning and genetics [3, 24]. Even though some recent works provided specific dependence of posterior convergence rates of topics on the number of documents [23, 39], the concrete dependence of these rates on the number of words in each document or the size of vocabulary has still been unknown. In these works, posterior convergence rates for parameters are established by using techniques previously used in analyzing densities. More precisely, the approach was based on deriving lower bounds on distances on density functions, including the Hellinger or total variation distances [13], in terms of distances between the model parameters. Given lower bounds of this type, a convergence guarantee for the density can immediately be translated into a convergence guarantee for the parameters. A drawback of this technique is that it typically requires strong global geometric assumptions that are not always necessary, especially for weakly-concave population log-likelihood functions, namely, the limit of sample log-likelihood function when the number of data n grows to infinity. For example, in order to guarantee, for some $\alpha \geq 1$, an $n^{-\frac{1}{2\alpha}}$ rate for parameter convergence, the bound in Ghosal et al. [13] requires the following conditions:

$$\begin{aligned} & -\mathbb{E}_{\theta^*}\left[\log \frac{p_\theta}{p_{\theta^*}}\right] \precsim \|\theta - \theta^*\|_2^{2\alpha}, \quad -\mathbb{E}_{\theta^*}\left[\left(\log \frac{p_\theta}{p_\theta^*}\right)^2\right] \precsim \|\theta - \theta^*\|_2^{2\alpha}, \quad \text{and} \\ & \|\theta_1 - \theta_2\|_2^{\alpha} \precsim h(p_{\theta_1}, p_{\theta_2}) \precsim \|\theta_1 - \theta_2\|_2^{\alpha}. \end{aligned} \tag{1}$$

Here θ^* denotes the true parameter, the quantity p_θ is the density function at θ, and h is the Hellinger distance. The conditions (1) do not always hold for the statistical models mentioned earlier, since the log-likelihood function behaves differently according to its location within the unbounded parameter space. Furthermore, the precise dependence of the convergence rate on other parameters in addition to sample size n, including the dimension d, can be difficult to obtain using this technique.

In this paper, we study the posterior contraction rates of parameters via the lens of diffusion processes. This approach does not require the parameter space to be bounded and allows us to quantify the dependence of convergence rates on quantities other than the sample size n. More specifically, we recast the posterior distribution as the stationary distribution of a stochastic differential equation (SDE). In this way, by controlling the moments of the given diffusion process, we can obtain bounds on posterior convergence rates of parameters. This approach exploits two main features that have not been extensively explored in the Bayesian literature: (i) the geometric structure of population log-likelihood function and (ii) uniform perturbation bounds between the empirical and population log-likelihood functions.

At a high-level, our main contributions can be summarized as follows:

- **Strongly concave setting:** We first consider settings in which the population log-likelihood function is strongly concave around the true parameter. We demonstrate that as long as the prior distribution is sufficiently smooth and the perturbation error between the population and empirical log-likelihood function is well-controlled, the posterior contraction rate around the true parameter is $(d/n)^{1/2}$. In addition, this technique allows us to also quantity the dependence of the rate on other properties of the model.

- **Weakly concave setting:** Moving beyond the strongly concave setting, we then turn to population log-likelihood functions that are only weakly concave. In this setting, our analysis depends on two auxiliary functions: (i) a function ψ is used to capture the local and global behavior of population log-likelihood function around the true parameter, and (ii) a function ζ describes the growth of perturbation error between the population and sample log-likelihood functions. Our analysis shows how the posterior contraction rates for parameters can be derived from the unique positive solution of a non-linear equation involving these two functions.

- **Illustrative examples:** We illustrate the general results for three concrete classes of models: Bayesian logistic regression models, Bayesian single index models, and over-specified Bayesian location Gaussian mixture models. The obtained rates show the influence of different modeling assumptions. For instance, the posterior convergence rate of parameter under Bayesian logistic regression models is $(d/n)^{1/2}$. In contrast, for Bayesian single index models with polynomial link functions, we exhibit rates of the form $(d/n)^{1/(2p)}$, where $p \geq 2$ is the degree of the polynomial. Finally, for over-specified location Gaussian mixtures, we establish a convergence rate of the order $(d/n)^{1/4}$. While the $n^{-1/4}$ component of this rate is known from past work [6, 22], the corresponding scaling of $d^{1/4}$ with dimension appears to be a novel result.

The remainder of the paper is organized as follows. In Section 2, we set up the basic framework for Bayesian models and introduce a diffusion process that admits posterior distribution as its stationary distribution. Section 3 is devoted to establishing the general results for posterior convergence rates of parameters under various assumptions on the concavity of the population log-likelihood. In Section 4, we apply these general results to derive concrete rates of convergence for a number of illustrative examples. The proofs of main theorems are provided in Section 5 while the proofs of auxiliary lemmas and corollaries in the paper are deferred to the Appendices. We conclude our work with a discussion in Section 6.

Notation. In the paper, the expression $a_n \gtrsim b_n$ will be used to denote $a_n \geq c b_n$ for some positive universal constant c that does not change with n. Additionally, we write $a_n \asymp b_n$ if both $a_n \gtrsim b_n$ and $a_n \lesssim b_n$ hold. For any $n \in \mathbb{N}$, we denote $[n] = \{1, 2, \ldots, n\}$. The notation

$\mathbb{S}^{d-1}$ stands for the unit sphere, namely, the set of vectors $u \in \mathbb{R}^d$ such that $\|u\|_2 = 1$. For any subset Θ of $\mathbb{R}^d$, $r \geq 1$, and $\varepsilon > 0$, we denote $\mathcal{N}(\varepsilon, \Theta, \|.\|_r)$ the covering number of Θ under $\|.\|_r$ norm, namely, the minimum number of ε-balls under $\|.\|_r$ norm to cover the entire set Θ. Finally, for any $x, y \in \mathbb{R}$, we denote $x \vee y = \max\{x, y\}$.

2 Background and problem formulation

We first provide formulation for posterior distribution of parameter and its convergence rate in Section 2.1. Then we demonstrate that the posterior distribution is a stationary distribution of a diffusion process in Section 2.2. Finally, we define the population likelihood function and provide smoothness conditions of that function and the prior distribution in Section 2.3.

2.1 Posterior contraction rates for parameters

Consider a parametric family of distributions $\{P_\theta \mid \theta \in \Theta\}$. Throughout the paper, we assume that each distribution P_θ has density p_θ with respect to the Lebesgue measure. Let $X_1^n := (X_1, \ldots, X_n)$ be a sequence of random variables drawn i.i.d. from P_{θ^*}, where $\theta^* \in \Theta$ is the true parameter, albeit unknown. The log-likelihood F_n of the data is given by

$$F_n(\theta) := \frac{1}{n} \sum_{i=1}^{n} \log p_\theta(X_i). \tag{2a}$$

In a Bayesian analysis, the parameter space Θ is endowed with a prior distribution π. Combining this prior with the likelihood (2a) yields the posterior distribution

$$\Pi\left(\theta \mid X_1^n\right) := \frac{e^{nF_n(\theta)}\pi(\theta)}{\int_\Theta e^{nF_n(u)}\pi(u)du}. \tag{2b}$$

As the sample size n increases, we expect that the posterior distribution will concentrate more of its mass over increasingly smaller neighborhoods of the true parameter θ^*. Posterior contraction rates allow us to study how quickly this concentration of mass takes place. In particular, for a given norm, we study the posterior mass of a ball of the form $\|\theta - \theta^*\| \leq \rho$ for a suitably chosen radius. For a given $\delta \in (0, 1)$, our goal is to prove statements of the form

$$\Pi\Big(\|\theta - \theta^*\| \geq \rho(n, d, \delta) \mid X_1^n\Big) \leq \delta, \tag{3}$$

with probability at least $1 - \delta$ over the randomly drawn data X_1^n. Our interest is in the scaling of the radius $\rho(n, d, \delta)$ as a function of sample size n, problem dimension d, and the error tolerance δ, as well as other problem-specific parameters.

2.2 From diffusion processes to the posterior distribution

The analysis of this paper relies on a connection between the posterior distribution and a particular stochastic differential equation (SDE). Accordingly, we begin by introducing some background on Langevin processes. For a parameter $\beta > 0$, consider an SDE of the form

$$d\theta_t = -\nabla U(\theta_t)dt + \sqrt{\frac{2}{\beta}}\, dB_t, \tag{4}$$

where $(B_t, t \geq 0)$ is a standard d-dimensional Brownian motion [25], and the potential function $U : \mathbb{R}^d \to \mathbb{R}$ is assumed to satisfy the regularity conditions: (a) its gradient ∇U is locally Lipschitz, and (b) its gradient satisfies the inequality

$$\langle \nabla U(\theta), \theta \rangle \geq c_1 \left\| \theta \right\|_2 - c_2 \qquad \text{for any } \theta \in \mathbb{R}^d,$$

where c_1, c_2 are positive constants. Under these conditions, from known results on general Langevin diffusions [20, 26], we have:

Proposition 1. *Under the stated regularity conditions, the solution to the Langevin SDE (4) exists and is unique in the strong sense. Furthermore, the density of θ_t converges in $\mathbb{L}^2$ to the stationary distribution with density proportional to $e^{-\beta U}$.*

Let us consider this general result in the context of Bayesian inference. In particular, suppose that we apply Proposition 1 to the potential function $U_n(\theta) := -nF_n(\theta) - \log \pi(\theta)$. Doing so will require us to verify that U_n satisfies the requisite regularity conditions. Assuming this validity, we are guaranteed that the posterior distribution $\Pi(\theta \mid X_1^n)$ is the stationary distribution of the following stochastic differential equation (SDE):

$$d\theta_t = \frac{1}{2} \nabla F_n(\theta_t) dt + \frac{1}{2n} \nabla \log \pi(\theta_t) dt + \frac{1}{\sqrt{n}} dB_t, \tag{5}$$

where $\theta_0 = \theta^*$. Moreover, the density of θ_t converges in $\mathbb{L}^2$ to the posterior density.

This connection—between the SDE (5) and the posterior distribution (2b)—provides an avenue for analysis. In particular, by characterizing the behavior of the process $(\theta_t, t \geq 0)$ as a function of time, we can obtain bounds on the posterior distribution by taking limits. In particular, Fatou's lemma guarantees that for any $p > 0$, we have

$$Z^{-1} \int \left\| \theta \right\|_2^p e^{-U_n(\theta)} d\theta \leq \liminf_{t \to +\infty} \mathbb{E}\left[\left\| \theta_t \right\|_2^p \right], \tag{6}$$

where $Z = \int e^{-U_n(\theta)} d\theta$. Given this bound, we can establish posterior contraction rates for the parameters by controlling the moments of the diffusion process $\{\theta_t\}_{t \geq 0}$. The main theoretical results of this paper are obtained by following this general roadmap.

2.3 From empirical to population likelihood

Before proceeding to our main results, let us introduce some additional definitions and conditions. A useful notion for our analysis is the population log-likelihood F. It corresponds to the limit of log-likelihood function F_n, as previously defined in equation (2a), as the sample size n goes to infinity—viz.

$$F(\theta) := \mathbb{E}\left[\log p_\theta(X) \right], \tag{7}$$

where the expectation is taken with respect to $X \sim P_{\theta^*}$. Throughout the paper, we impose the following smoothness conditions on the population log-likelihood F and the log prior density:

(A) There exist positive constants L_1 and L_2 such that for any $\theta_1, \theta_2 \in \mathbb{R}^d$, we have

$$\left\| \nabla F(\theta_1) - \nabla F(\theta_2) \right\|_2 \leq L_1 \left\| \theta_1 - \theta_2 \right\|_2,$$
$$\left\| \nabla \log \pi(\theta_1) - \nabla \log \pi(\theta_2) \right\|_2 \leq L_2 \left\| \theta_1 - \theta_2 \right\|_2.$$

(B) There exists a constant $B > 0$ such that

$$\sup_{\theta \in \mathbb{R}^d} \langle \nabla \log \pi(\theta),\, \theta - \theta^* \rangle \leq B.$$

Note that, the constant B in Assumption B depends on θ^*. We suppress that dependence in B for the simplicity of the presentation. The above conditions are relatively mild, and we provide a number of examples in the sequel for which they are satisfied.

3 Main results

We now turn to the presentation of our main results, which provides guarantees on posterior contraction rates under different conditions. Our first main result, stated as Theorem 1 in Section 3.1, establishes the posterior convergence rate of parameters when the population log-likelihood is strongly concave around the true parameter θ^*. Then in Section 3.2, we study the same question when the population log-likelihood is weakly concave around θ^*, with our conclusions stated in Theorem 2.

3.1 Posterior contraction under strong concavity

We first study the setting when F is strongly concave around the true parameter θ^*. We collect a few assumptions that are needed for the analysis:

(S.1) There exists a scalar $\mu > 0$ such that

$$\langle \nabla F(\theta),\, \theta^* - \theta \rangle \geq \mu \left\|\theta - \theta^*\right\|_2^2 \quad \text{for any } \theta \in \mathbb{R}^d. \tag{8}$$

(S.2) For any $\delta > 0$, there exist non-negative functions ε_1 and ε_2 that map from $\mathbb{N} \times (0, 1]$ to $\mathbb{R}_+$ such that

$$\sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\|\nabla F_n(\theta) - \nabla F(\theta)\right\|_2 \leq \varepsilon_1(n, \delta) r + \varepsilon_2(n, \delta),$$

for any radius $r > 0$ with probability at least $1 - \delta$.

Assumption S.1 is a standard strong concavity condition of function F around θ^*. Furthermore, Assumption S.2 is used to control the uniform perturbation error between the gradient of negative empirical log-likelihood function F_n and the gradient of negative population log-likelihood function F.

Given the above assumptions, we are ready to state our first result regarding the posterior convergence rate of parameters for a strongly concave population log likelihood:

Theorem 1. *Suppose that Assumptions A, B, S.1, and S.2 hold. Then there is a universal constant C such that for any $\delta \in (0, 1)$, given a sample size n large enough such that $\varepsilon_1(n, \delta) \leq \frac{\mu}{6}$, we have*

$$\Pi\left(\left\|\theta - \theta^*\right\|_2 \geq C\sqrt{\frac{d + \log(1/\delta) + B}{n\mu}} + \frac{\varepsilon_2(n, \delta)}{\mu} \;\Big|\; X_1^n \right) \leq \delta,$$

with probability $1 - \delta$, taken with respect to the random observations X_1^n.

See Section 5.1 for the proof of Theorem 1.

The result of Theorem 1 establishes the posterior convergence rate $(d/n)^{1/2}$ of parameter under the strong concavity settings of F. It also provides a detailed dependence of the rate on other model parameters, including B and μ, both of which might vary as a function of θ^*. At the moment, we do not know whether the dependence of these parameters is optimal.

3.2 Posterior contraction under weak concavity

Moving beyond the strong concavity setting, we consider the weakly concave setting of the population log-likelihood function F. Weak concavity arises when working with singular statistical models, meaning those whose Fisher information matrix at the true parameter θ^* is rank-degenerate. Examples of such models include single index models [21] with certain choices of link functions, as well as over-specified Bayesian mixture models [28], in which the fitted mixture model has more components than the true mixture distribution. In order to analyze the posterior contraction rate of parameters for weakly concave log likelihoods, we impose the following assumptions:

(W.1) There exists a convex, non-decreasing function $\psi : [0, +\infty) \to \mathbb{R}$ such that $\psi(0) = 0$ and for any $\theta \in \mathbb{R}^d$, we have

$$\langle \nabla F(\theta),\, \theta^* - \theta \rangle \geq \psi(\|\theta - \theta^*\|_2).$$

Assumption W.1 characterizes the weak concavity of the function F around the global maxima θ^*. This condition can hold when the log likelihood is locally strongly concave around θ^* but only weakly concave in a global sense, or it can hold when the log likelihood is weakly concave but nowhere strongly concave. An example of the former type is the logistic regression model analyzed in Section 4.1, whereas an example of the latter type is given by certain kinds of single index models, as analyzed in Section 4.2.

Our next assumption controls the deviation between the gradients of the population and sample likelihoods:

(W.2) For any $\delta > 0$, there exist a function $\varepsilon : \mathbb{N} \times (0, 1] \mapsto \mathbb{R}_+$ and a non-decreasing function $\zeta : \mathbb{R} \to \mathbb{R}$ such that $\zeta(0) \geq 0$ and

$$\sup_{\theta \in \mathbb{B}(\theta^*, r)} \|\nabla F_n(\theta) - \nabla F(\theta)\|_2 \leq \varepsilon(n, \delta)\zeta(r),$$

for any radius $r > 0$ with probability at least $1 - \delta$.

Note that, the function ζ can depend on the sample size n and other model parameters, as analyzed in Section 4.3 under over-specified Bayesian mixture model. Here we suppress this dependence in ζ for the brevity of the presentation.

The previous conditions involved two functions, namely ψ and ζ. We let $\xi : \mathbb{R}_+ \to \mathbb{R}$ denote the inverse function of the strictly increasing function $r \mapsto r\zeta(r)$. Our next assumption imposes certain inequalities on these functions and their derivatives:

(W.3) The function $r \mapsto \psi(\xi(r))$ is convex, and moreover, for any $r > 0$, the functions ψ and ζ satisfy the following differential inequalities:

$$\begin{aligned} r\psi'(r)\zeta(r) &\geq r\psi(r)\zeta'(r) + \psi(r)\zeta(r), \quad \text{and} \\ r^2\psi''(r)\zeta(r) + r\psi'(r)\zeta(r) &\geq 3\psi(r)\zeta(r) + r^2\psi(r)\zeta''(r). \end{aligned}$$

These differential inequalities are needed controlling the moments of the diffusion process $\{\theta_t\}_{t>0}$ in equation (5). In our discussion of concrete examples, we provide instances in which they are satisfied.

Our result involves a certain fixed point equation that depends on the parameters and functions in our assumptions. In particular, for any tolerance parameter $\delta \in (0,1)$ and sample size n, consider the following equation in a variable $z > 0$:

$$\psi(z) = \varepsilon(n,\delta)\zeta(z)z + \frac{B + d\log(1/\delta)}{n}. \tag{9}$$

In order to ensure that equation (9) has a unique positive solution, our final assumption imposes certain condition on the growth of the functions ψ and ζ:

(W.4) The sample size n and tolerance parameter $\delta \in (0,1)$ are such that $\varepsilon(n,\delta) < \lim\inf_{z\to+\infty} \frac{\psi(z)}{z\zeta(z)}$.

With this set-up, we are now ready to state our second main result:

Theorem 2. *Assume that Assumptions A, B, and W.1—W.4 hold. Then for a given sample size n and $\delta \in (0,1)$, equation (9) has a unique positive solution $z^*(n,\delta)$ such that*

$$\Pi\Big(\|\theta - \theta^*\|_2 \geq z^*(n,\delta) \ \mid \ X_1^n\Big) \leq \delta \tag{10}$$

with probability $1-\delta$ with respect to the random observations X_1^n.

See Section 5.2 for the proof of Theorem 2.

A few comments are in order. First, the convergence guarantee (10) depends on the weak convexity function ψ and the perturbation function ζ through the non-linear equation (9). It is natural to wonder about the origins of this equation. As shown in our analysis, it stems from an upper bound on the moments $\mathbb{E}\left[\|\theta_t - \theta^*\|_2^p\right]$ of the diffusion process $\{\theta_t\}_{t>0}$ defined in equation (5). In particular, we find that

$$\lim_{t\to+\infty} \left(\mathbb{E}\left(\|\theta_t - \theta^*\|_2^p\right)\right)^{\frac{1}{p}} \leq z_p^*,$$

where z_p^* is the unique positive solution to the equation $\psi(z) = \varepsilon(n,\delta)\zeta(z)z + \frac{B+pd}{n}$. In light of the above result and the inequality (6), when p is of the order $\log(1/\delta)$, we obtain the posterior convergence rate (10).

Second, it is (at least in general) not possible to compute an explicit form for the positive solution $z^*(n,\delta)$ to the non-linear equation (9). However, for certain forms of the function ψ and ζ, it is possible to compute an analytic and relatively simple upper bound. For instance, given some positive parameters (α,β) such that $\alpha > \beta + 1$, suppose that these functions are defined locally, in a interval above zero, as follows:

$$\psi(r) = r^\alpha, \quad \text{and} \quad \zeta(r) = r^\beta \qquad \text{for all } r \text{ in some interval } [0,\bar{r}). \tag{11a}$$

Moreover, suppose that the perturbation function takes the form

$$\varepsilon(n,\delta) = \sqrt{\left(d + \log(\tfrac{1}{\delta})\right)/n}. \tag{11b}$$

As shown in the analysis to follow in Section 4, these particular forms arise in several statistical models, including Bayesian logistic regression and certain forms of Bayesian single index models. The following result shows that we have a simple upper bound on $z^*(n,\delta)$.

Corollary 1. *Assume that the functions ψ, ζ have the local behavior (11a), and the perturbation term $\varepsilon(n,\delta)$ has the form (11b). If, in addition, the global forms of ψ and ζ satisfy Assumption W.3, then we are guaranteed that*

$$z^*(n,\delta) \leq c \left(\frac{d+\log(1/\delta)+B}{n}\right)^{\frac{1}{2(\alpha-(\beta+1))}} \vee \left(\frac{d+\log(1/\delta)+B}{n}\right)^{\frac{1}{\alpha}},$$

where $z^*(n,\delta)$ is defined in Theorem 2.

Corollary 1 directly leads to a posterior convergence rate for the parameters—namely, we have

$$\Pi\left(\|\theta-\theta^*\|_2 \geq c\left(\frac{d+\log(1/\delta)+B}{n}\right)^{\frac{1}{2(\alpha-(\beta+1))}} \vee \left(\frac{d+\log(1/\delta)+B}{n}\right)^{\frac{1}{\alpha}} \;\middle|\; X_1^n\right) \leq \delta, \quad (12)$$

with probability $1-\delta$ with respect to the training data. Note that the convergence rate scales as $(d/n)^{\frac{1}{2(\alpha-(\beta+1))}}$ when $\alpha \geq 2(\beta+1)$. On the other hand, this rate becomes $(d/n)^{\frac{1}{2\alpha}}$ when $\alpha < 2(\beta+1)$.

4 Some illustrative examples

In this section, we study the posterior contraction rates of parameters for a few interesting statistical examples that fall under the general framework of this paper.

4.1 Bayesian logistic regression

We begin with the method of logistic regression, which is popular approach to modeling the relation between a binary response variable $Y \in \{-1,+1\}$ and a vector $X \in \mathbb{R}^d$ of explanatory variables [21]. In this model, the pair Y and X are related by a conditional distribution of the form

$$\mathbb{P}(Y=1 \mid X,\theta) = \frac{e^{\langle X,\theta\rangle}}{1+e^{\langle X,\theta\rangle}}, \quad (13)$$

where $\theta \in \mathbb{R}^d$ is a parameter vector.

Suppose that we observe a collection $Z_1^n = \{Z_i\}_{i=1}^n$ of n i.i.d paired samples $Z_i = (X_i, Y_i)$, generated in the following way. First, the covariate vector X_i is drawn from a standard Gaussian distribution $N(0, I_d)$, and then the binary response Y_i is drawn according to the conditional distribution:

$$\mathbb{P}(Y_i=1 \mid X_i,\theta^*) = \frac{e^{\langle X_i,\theta^*\rangle}}{1+e^{\langle X_i,\theta^*\rangle}}, \quad (14)$$

where $\theta^* \in \mathbb{R}^d$ is a fixed but unknown value of the parameter vector. Consequently, the sample log-likelihood function of the samples Z_1^n takes the form

$$F_n^R(\theta) := \frac{1}{n}\sum_{i=1}^n \left\{\log \mathbb{P}(Y_i \mid X_i,\theta) + \log \phi(X_i)\right\}, \quad (15)$$

where ϕ denotes the density of a standard normal vector. Combining this log likelihood with a given prior π over θ yields the posterior distribution in the usual way. We assume that the prior function π satisfies Assumptions A and B.

With this set-up, the following result establishes the posterior convergence rate of θ around θ^*, conditionally on the observations Z_1^n.

Corollary 2. *For a given $\delta \in (0,1)$, suppose that we are given $\frac{n}{\log n} \geq c' d \log(\frac{1}{\delta})$ i.i.d. samples Z_1^n from the Bayesian logistic regression model (13) with true parameter θ^* for some universal constant c'. Then there is a universal constant c such that*

$$\Pi\left(\|\theta - \theta^*\|_2 \geq c\sqrt{\frac{d + \log(1/\delta) + B}{n}} \;\middle|\; Z_1^n \right) \leq \delta$$

with probability $1-\delta$ over the data Z_1^n.

See Appendix A.1 for the proof of this claim.

A few comments are in order. First, the result of Corollary 2 shows that the posterior convergence rate of parameters under Bayesian logistic regression model (13) is $(d/n)^{1/2}$. Furthermore, this result also gives a concrete dependence of the rate on B characterizing the degree to which the prior is concentrated away from the true parameter.

Second, by taking the sample size in the function F_n^R to infinity, the population log-likelihood is given by

$$F^R(\theta) := \mathbb{E}_{(X,Y)}\left[-\log\left(1 + e^{-Y\langle X, \theta\rangle}\right) + \log \phi(X) \right], \tag{16}$$

where the outer expectation in the above display is taken with respect to X and $Y|X$ from the logistic model (14).

In Appendix A.1, we prove that there are universal constants c, c_1, c_2 such that

$$\langle \nabla F^R(\theta), \theta^* - \theta\rangle \geq c_1 \begin{cases} \|\theta - \theta^*\|_2^2, & \text{for all } \|\theta - \theta^*\|_2 \leq 1 \\ \|\theta - \theta^*\|_2, & \text{otherwise} \end{cases} \tag{17a}$$

and

$$\sup_{\theta \in \mathbb{R}^d} \left\| \nabla F_n^R(\theta) - \nabla F^R(\theta) \right\|_2 \leq c_2 \left(\sqrt{\frac{d}{n}} + \sqrt{\frac{\log(1/\delta)}{n}} + \frac{\log(1/\delta)}{n} \right), \tag{17b}$$

for any $r > 0$ with probability $1-\delta$ as long as $\frac{n}{\log n} \geq cd\log(1/\delta)$. The above results with F^R and F_n^R indicate that the functions ψ and ζ in Assumptions W.1 and W.2 have the following close forms:

$$\psi(r) = c_1 \begin{cases} r^2, & \text{for all } 0 < r \leq 1 \\ r, & \text{otherwise} \end{cases}, \quad \text{and} \quad \zeta(r) = c_2 \quad \text{for all } r > 0. \tag{18}$$

We can check that the functions ψ and ζ satisfy the conditions in Assumptions W.3 and W.4. Therefore, an application of Theorem 2 to these functions leads to the posterior contraction rate of θ around θ^* in Corollary 2.

4.2 Bayesian single index models

We now turn to single index models, which are generalizations of linear regression models where the linear coefficients are connected to a general link function [5]. These models are widely used in both econometrics and biostatistics, and have also seen application in computational imaging problems. Due to the special structure of single index models, curse of dimensionality associated with model index coefficients can be avoided. In this section, we consider specific settings of single index models when the link function is known and has specific form. In particular, we assume that the data points $Z_i = (Y_i, X_i) \in \mathbb{R}^{d+1}$ are generated as follows:

$$Y_i = g(X_i^\top \theta^*) + \epsilon_i, \tag{19}$$

for $i \in [n]$ where the positive integer $p \geq 2$ is given. Here (X_i, Y_i) correspond to the covariate vector and response variable, respectively, whereas g is a *known* link function while θ^* is a true but unknown parameter. Furthermore, the random variables $\{\epsilon_i\}_{i=1}^n$ are i.i.d. standard Gaussian variables, whereas the covariates X_i are assumed to be i.i.d. data from standard multivariate Gaussian distribution. The choice $g(r) = r^2$ has been used to model the problem of phase retrieval in computational imaging.

Now, we consider a Bayesian single index model to study θ^*. More specifically, we endow the parameter θ with a prior function π. Conditioning on X_i and θ, we have

$$Y_i \mid X_i, \theta \overset{\text{i.i.d.}}{\sim} \mathcal{N}\left(g\left(X_i^\top \theta\right), 1\right). \tag{20}$$

The main goal of this section is to study the posterior convergence rate of θ around θ^* given the data. To do that, we assume the prior function π satisfies the Assumptions A and B and follow the framework in the Bayesian logistic regression case. In particular, we first study the structure of the sample log-likelihood function around the true parameter θ^*. Then we establish the uniform perturbation bound between the population and sample log-likelihood functions.

Given the Bayesian single index model (20), the sample log-likelihood function F_n^I of the samples $Z_1^n = \{Z_i\}_{i=1}^n$ admits the following form

$$F_n^I(\theta) := \frac{1}{n}\left(\sum_{i=1}^n -\frac{\left(Y_i - g\left(X_i^\top \theta\right)\right)^2}{2} + \log \phi(X_i)\right), \tag{21}$$

where ϕ is the standard normal density function of $X_1, \ldots, X_n$. Hence, the population log-likelihood function F^I has the following form

$$F^I(\theta) := \mathbb{E}_{(X,Y)}\left[-\frac{\left(Y - g\left(X^\top \theta\right)\right)^2}{2} + \log \phi(X)\right], \tag{22}$$

where the outer expectation in the above display is taken with respect to $X \sim \mathcal{N}(0, I_d)$ and $Y|X = x \sim \mathcal{N}\left(g\left(x^\top \theta^*\right), 1\right)$.

We can check that the function F^I is weak-concave around θ^* when the link function g and the true parameter θ^* take the following values

$$g(r) = r^p \quad \text{for some } p \geq 2, \text{ and} \quad \theta^* = 0. \tag{23}$$

Given these choices of g and θ^*, the population log- likelihood function has the closed-form expression

$$F^I(\theta) = \frac{1 + (2p-1)!! \, \|\theta - \theta^*\|_2^{2p}}{2} \qquad \text{for all } \theta \in \mathbb{R}^d.$$

Furthermore, in Appendix A.2, we prove that there is a universal constant $c_1 > 0$ such that

$$\langle \nabla F^I(\theta), \, \theta^* - \theta \rangle \geq c_1 \, \|\theta - \theta^*\|_2^{2p} \quad \text{for all } \theta \in \mathbb{R}^d, \tag{24a}$$

and there are universal constants (c, c_2) such that for any $r > 0$ and $\delta \in (0,1)$, as long as $n \geq c\,(d \log(d/\delta))^{2p}$, we have

$$\sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \nabla F_n^I(\theta) - \nabla F^I(\theta) \right\|_2 \leq c_2 \left(r^{p-1} + r^{2p-1} \right) \sqrt{\frac{d + \log(1/\delta)}{n}}, \tag{24b}$$

with probability at least $1 - \delta$. Therefore, the functions ψ and ζ in Assumptions W.1 and W.2 take the specific forms

$$\psi(r) = c_1 r^{2p}, \quad \text{and} \quad \zeta(r) = r^{p-1} + r^{2p-1}, \tag{25}$$

for all $r > 0$. Simple algebra shows that these functions satisfy Assumptions W.3 and W.4. Therefore, under the setting (23), a direct application of Theorem 2 leads to the following result regarding the posterior contraction rate of θ around θ^*:

Corollary 3. *Consider the Bayesian single index model (19) with true parameter $\theta^* = 0$ and link function $g(r) = r^p$ for for some $p \geq 2$. Then there are universal constants c, c' such that for any $\delta > 0$, given a sample size $n \geq c'(d + \log(d/\delta))^{2p}$, we have*

$$\Pi\left(\|\theta - \theta^*\|_2 \geq c \left(\frac{d + \log(1/\delta) + B}{n} \right)^{1/(2p)} \;\middle|\; Z_1^n \right) \leq \delta$$

with probability $1 - \delta$ over the data Z_1^n.

See Appendix A.2 for the proof of Corollary 3.

It is worth noting that the proof of Corollary 3 actually leads to the following stronger uniform perturbation bound:

$$\begin{aligned} \sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \nabla F_n^I(\theta) - \nabla F^I(\theta) \right\|_2 \leq c\, r^{p-1} & \left(\sqrt{\frac{d + \log \frac{1}{\delta}}{n}} + \frac{1}{n^{3/2}} \left(d + \log \frac{n}{\delta} \right)^{p+1} \right) \\ & + r^{2p-1} \left(\sqrt{\frac{d + \log \frac{1}{\delta}}{n}} + \frac{1}{n^{3/2}} \left(d + \log \frac{n}{\delta} \right)^{2p+1} \right), \end{aligned}$$

valid for each $r > 0$ with probability $1 - \delta$. The condition $n \geq c(d + \log(d/\delta))^{2p}$ is required to guarantee that the RHS of the above display is upper bounded by the RHS of equation (24b); this bound permits us to apply Theorem 2 to establish the posterior convergence rate of parameter under the Bayesian single index models.

4.3 Over-specified Bayesian Gaussian mixture models

Bayesian Gaussian mixture models have been widely used by statisticians to study datasets with heterogeneity, which can be modeled in terms of different mixture components [19]. In fitting such models, the true number of components is generally unknown, and several approaches have been proposed to deal with this challenge. One of the most popular methods is to overspecify the true number of components and fit the data with the larger models. This method is regarded as overspecified Gaussian mixture models [28]. Posterior convergence rates of density function with over-specified Bayesian Gaussian mixture models have been well-studied [14]. However, rates of parameters in these models are not fully understood. On one hand, when the covariance matrices are known and the parameter space is bounded, location parameters have been shown to have posterior convergence rates $n^{-1/4}$ in the Wasserstein-2 metric [22]. However, neither the dependence on dimension d nor on the true number of components have been established.

In this section, we consider a class of overspecified location Gaussian mixture models and provide convergence rates for the parameters with precise dependence on the dimension d, and not requiring any boundedness assumption on the parameter space. Concretely, suppose that $X_1^n = (X_1, \ldots, X_n)$ are i.i.d. samples from single Gaussian distribution $\mathcal{N}(\theta^*, I_d)$ where $\theta^* = 0$. Suppose that we fit such a dataset using an overspecified Bayesian location Gaussian mixture model of the form

$$\theta \sim \pi(\theta), \qquad c_i \in \{-1, 1\} \overset{\text{i.i.d.}}{\sim} \text{Cat}(1/2, 1/2), \qquad X_i | c_i, \theta \overset{\text{i.i.d.}}{\sim} \mathcal{N}(c_i \theta, I_d), \tag{26}$$

where $\text{Cat}(1/2, 1/2)$ stands for the categorical distribution with parameters $(1/2, 1/2)$. Here the prior function π is chosen to satisfy smoothness Assumptions A and B, with one example being the location Gaussian density function. Our goal in this section is to characterize the posterior contraction rate of the location parameter θ around θ^*.

In order to do so, we first define the sample log-likelihood function F_n^G given data X_1^n. It has the form

$$F_n^G(\theta) := \frac{1}{n} \sum_{i=1}^{n} \log \left(\frac{1}{2} \phi(X_i; -\theta, I_d) + \frac{1}{2} \phi(X_i; \theta, I_d) \right), \tag{27}$$

where $x \mapsto \phi(x; \theta, I_d) = (2\pi)^{-d/2} e^{-\|x-\theta\|_2^2/2}$ denotes the density of multivariate Gaussian distribution $\mathcal{N}(\theta, \sigma^2 I_d)$. Similarly, the population log-likelihood function is given by

$$F^G(\theta) := \mathbb{E}_X \left[\log \left(\frac{1}{2} \phi(X; -\theta, I_d) + \frac{1}{2} \phi(X; \theta, I_d) \right) \right], \tag{28}$$

where the outer expectation in the above display is taken with respect to $X \sim \mathcal{N}(\theta^*, I_d)$.

In Appendix A.3, we prove that there is a universal constant $c_1 > 0$ such that

$$\langle \nabla F^G(\theta), \, \theta^* - \theta \rangle \geq \begin{cases} c_1 \|\theta - \theta^*\|_2^4, & \text{for all } \|\theta - \theta^*\|_2 \leq \sqrt{2} \\ 4c_1 \left(\|\theta - \theta^*\|_2^2 - 1 \right), & \text{otherwise} \end{cases}, \tag{29a}$$

and moreover, there are universal constants (c, c_2) such that for any $\delta \in (0, 1)$, given a sample size $n \geq cd \log(1/\delta)$, we have

$$\sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \nabla F_n^G(\theta) - \nabla F^G(\theta) \right\|_2 \leq c_2 \left(r + \frac{1}{\sqrt{n}} \right) \sqrt{\frac{d + \log(\log n/\delta)}{n}}. \tag{29b}$$

with probability $1 - \delta$.

Given the above results, the functions ψ and ζ in Assumptions W.1 and W.2 take the form

$$\psi(r) = \begin{cases} c_1 r^4, & \text{for all } 0 < r \leq \sqrt{2} \\ 4c_1 \left(r^2 - 1 \right), & \text{otherwise} \end{cases}, \quad \text{and} \quad \zeta(r) = r + \frac{1}{\sqrt{n}} \quad \text{for all } r > 0. \tag{30}$$

These functions satisfy the conditions of Assumptions W.3 and W.4. Therefore, it leads to the following result regarding the posterior contraction rate of parameters under overspecified Bayesian location Gaussian mixtures (26):

Corollary 4. *Given the overspecified Bayesian location Gaussian mixture model* (26)*, there are universal constants c, c' such that given any $\delta \in (0,1)$ and a sample size $n \geq c' d \log(1/\delta)$, we have*

$$\Pi\left(\|\theta - \theta^*\|_2 \geq c \left(\frac{d + \log(\log n/\delta) + B}{n} \right)^{1/4} \;\Big|\; X_1^n \right) \leq \delta$$

with probability $1-\delta$ over the data X_1^n.

See Appendix A.3 for the proof of Corollary 4.

The dependence on n in the posterior contraction rate of θ in Corollary 4 is consistent with the previous result with location parameters in the overspecified Bayesian location Gaussian mixtures [6, 17, 22]. The novel finding in the result of the above corollary is the dependency on dimension d, which is $d^{1/4}$, and the dependency on the smoothness of the prior distribution π, which is $B^{1/4}$. Finally, our result does not require the boundedness of the parameter space, which had been a main assumption in previous works to study the posterior rate of θ [6, 17, 22].

5 Proofs

This section is devoted to the proofs of Theorem 1 and Theorem 2, with more technical aspects deferred to the appendices.

5.1 Proof of Theorem 1

Throughout the proof, in order to simplify notation, we omit the conditioning on the σ-field $\mathcal{F}_n := \sigma(X_1^n)$; it should be taken as given. For $\alpha = \frac{1}{2}\mu - \varepsilon_1(n,\delta) > \frac{\mu}{6}$, we claim that

$$\frac{1}{2} e^{\alpha t} \|\theta_t - \theta^*\|_2^2 \leq \frac{1}{\sqrt{n}} M_t + U_n \frac{(e^{\alpha t} - 1)}{2\alpha}, \tag{31}$$

where $U_n := \frac{B}{n} + \frac{3\varepsilon_2^2(n,\delta)}{\mu} + \frac{d}{n}$ and $M_t := \int_0^t e^{\alpha s} \langle \theta_s - \theta^*,\, dB_s \rangle$, which is a martingale.

Assume that the above claim is given at the moment (the proof of that claim is deferred to the end of the proof of the theorem). In order to bound the moments of martingale M_t, for any $p \geq 4$, we invoke the Burkholder-Gundy-Davis inequality [25] to find that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0 \leq t \leq T} |M_t|^{\frac{p}{2}} \right] &\leq (pC)^{\frac{p}{4}} \mathbb{E}\left[\langle M,\, M \rangle_T^{\frac{p}{4}} \right] = (pC)^{\frac{p}{4}} \mathbb{E}\left(\int_0^T e^{2\alpha s} \|\theta_s - \theta^*\|_2^2 \, ds \right)^{\frac{p}{4}} \\
&\leq (pC)^{\frac{p}{4}} \mathbb{E}\left(\sup_{0 \leq t \leq T} e^{\alpha t} \|\theta_t - \theta^*\|_2^2 \int_0^T e^{\alpha s} ds \right)^{\frac{p}{4}} \\
&\leq \left(\frac{pCe^{\alpha T}}{\alpha} \right)^{\frac{p}{4}} \mathbb{E}\left(\sup_{0 \leq t \leq T} e^{\alpha t} \|\theta_s - \theta^*\|_2^2 \right)^{\frac{p}{4}},
\end{aligned}$$

where C is a universal constant. Therefore, we arrive at the following bound:

$$\begin{aligned}
\mathbb{E}\left[e^{\alpha t} \|\theta_t - \theta^*\|_2 \right]^p &\leq \mathbb{E}\left(\frac{2}{\sqrt{n}} M_t \right)^{\frac{p}{2}} + \left(U_n \frac{(e^{\alpha t} - 1)}{\alpha} \right)^{\frac{p}{2}} \\
&\leq \left(U_n \frac{e^{\alpha T}}{\alpha} \right)^{\frac{p}{2}} + \left(\frac{pCe^{\alpha T}}{\alpha n} \right)^{\frac{p}{4}} \mathbb{E}\left(\sup_{0 \leq s \leq T} e^{\alpha s} \|\theta_s - \theta^*\|_2^2 \right)^{\frac{p}{4}}.
\end{aligned}$$

For the right hand side of the above inequality, we can relate it to the left hand side by using Young's inequality, which is given by

$$\left(\frac{pCe^{\alpha T}}{\alpha n}\right)^{\frac{p}{4}} \mathbb{E}\left(\sup_{0\le s\le T} e^{\alpha s}\left\|\theta_s-\theta^*\right\|_2^2\right)^{\frac{p}{4}} \le \frac{1}{2}\left(\frac{pCe^{\alpha T}}{\alpha n}\right)^{\frac{p}{2}} + \frac{1}{2}\mathbb{E}\left(\sup_{0\le s\le T} e^{\alpha s}\left\|\theta_s-\theta^*\right\|_2^2\right)^{\frac{p}{2}}.$$

Putting the above results together, we find that

$$\left(\mathbb{E}\left[\|\theta_T-\theta^*\|_2^p\right]\right)^{\frac{1}{p}} \le e^{-\alpha T}\left(\mathbb{E}\sup_{0\le t\le T}\left(e^{\alpha T}\left\|\theta_T-\theta^*\right\|_2^p\right)\right)^{\frac{1}{p}} \le C'\left(\sqrt{\frac{U_n}{\mu}}+\sqrt{\frac{2p}{n\mu}}\right),$$

for universal constant $C' > 0$. Therefore, the diffusion process defined in equation (5) satisfies the following inequality

$$\sup_{t\ge 0}\left(\mathbb{E}\left[\|\theta_t-\theta^*\|_2^p\right]\right)^{\frac{1}{p}} \le c\left(\sqrt{\frac{B+d}{\mu n}}+\frac{\varepsilon_2(n,\delta)}{\mu}+\sqrt{\frac{p}{n\mu}}\right)$$

for any $p \ge 1$. Combining the above inequality with the inequality (6) yields the conclusion of the theorem.

Proof of claim (31): For the given choice $\alpha > 0$, an application of Itô's formula yields the decomposition

$$\begin{aligned}
\frac{1}{2}e^{\alpha t}\left\|\theta_t-\theta^*\right\|_2^2 = &-\frac{1}{2}\int_0^t \langle\theta^*-\theta_s,\, \nabla F_n(\theta_s)e^{\alpha s}\rangle ds + \frac{1}{2n}\int_0^t \langle\theta_s-\theta^*,\, \nabla\log\pi(\theta_s)e^{\alpha s}\rangle ds \\
&+\frac{d}{2n}\int_0^t e^{\alpha s}ds + \frac{1}{\sqrt{n}}\int_0^t e^{\alpha s}\langle\theta_s-\theta^*,\, dB_s\rangle + \frac{1}{2}\int_0^t \alpha e^{\alpha s}\left\|\theta_s-\theta^*\right\|_2^2 ds \\
=&J_1+J_2+J_3+J_4+J_5.
\end{aligned} \tag{32}$$

We begin by bounding the term J_1 in equation (32). Based on the Assumption S.2 regarding the perturbation error between F_n and F and the strong convexity of F, we have

$$\begin{aligned}
J_1 &= -\frac{1}{2}\int_0^t \langle\theta^*-\theta_s,\, \nabla F_n(\theta_s)e^{\alpha s}\rangle ds \\
&\le -\frac{1}{2}\int_0^t \langle\theta^*-\theta_s,\, \nabla F(\theta_s)e^{\alpha s}\rangle ds + \frac{1}{2}\int_0^t \left\|\theta_s-\theta^*\right\|_2 \left\|\nabla F(\theta_s)-\nabla F_n(\theta_s)\right\|_2 e^{\alpha s}ds \\
&\le -\frac{1}{2}\int_0^t \mu\left\|\theta_s-\theta^*\right\|_2^2 e^{\alpha s}ds + \frac{1}{2}\int_0^t \left\|\theta_s-\theta^*\right\|_2 \left(\varepsilon_1(n,\delta)\left\|\theta_s-\theta^*\right\|_2+\varepsilon_2(n,\delta)\right)e^{\alpha s}ds \\
&\le -\frac{1}{2}\int_0^t \mu\left\|\theta_s-\theta^*\right\|_2^2 e^{\alpha s}ds + \frac{1}{2}\int_0^t \left\|\theta_s-\theta^*\right\|_2^2 \left(\varepsilon_1(n,\delta)+\mu/3\right)e^{\alpha s}ds + \frac{3\varepsilon_2^2(n,\delta)}{2\mu}\int_0^t e^{\alpha s}ds.
\end{aligned}$$

The second term J_2 involving prior π can be controlled in the following way:

$$J_2 = \frac{1}{2n}\int_0^t \langle\theta_s-\theta^*,\, \nabla\log\pi(\theta_s)e^{\alpha s}\rangle ds \le \frac{1}{2n}\int_0^t B\cdot e^{\alpha s}ds = \frac{(e^{\alpha t}-1)B}{2\alpha n}.$$

For the third term J_3, a direct calculation leads to

$$J_3 = \frac{d(e^{\alpha t}-1)}{2\alpha n}.$$

Moving to the fourth term J_4, it is a martingale as $J_4 = M_t/\sqrt{n}$. Putting the above results together, as $\alpha = \frac{1}{2}\mu - \varepsilon_1(n,\delta) > \frac{\mu}{6}$, we obtain that

$$\frac{1}{2}e^{\alpha t}\left\|\theta_t - \theta^*\right\|_2^2 \leq \frac{1}{\sqrt{n}}M_t + U_n\frac{(e^{\alpha t}-1)}{2\alpha}.$$

Putting together the pieces yields the claim (31).

5.2 Proof of Theorem 2

As in the proof of Theorem 1, we omit the conditioning on $\mathcal{F}_n := \sigma(X_1^n)$. For any $p \geq 2$, we define the function $\nu_{(p)}$ as $\nu_{(p)}(r) := \psi\left(r^{\frac{1}{p-1}}\right) r^{\frac{p-2}{p-1}}$ for any $r > 0$. Additionally, the function $\tau_{(p)}$ is defined to satisfy $\tau_{(p)}(r^{p-1}\zeta(r)) := r^{p-2}\psi(r)$ for any $r > 0$. Note that, due to Assumption W.2, the function $r \mapsto r^{p-1}\zeta(r)$ is a strictly increasing and surjective function that maps from $[0,+\infty)$ to $[0,+\infty)$. Therefore, it is invertible and the function $\tau_{(p)}^{-1}$ is well-defined.

Now, we claim that the one-dimensional functions $\nu_{(p)}(\cdot)$ and $\tau_{(p)}(\cdot)$ are convex and strictly increasing for any $p \geq 2$. Furthermore, the following inequality holds:

$$\mathbb{E}\left[\|\theta_t - \theta^*\|_2^p\right] \leq \frac{p}{2}\int_0^t \Bigg(- R_p(s) + \varepsilon(n,\delta)\tau_{(p)}^{-1}(R_p(s)) \\ + \frac{B+(p-1)d}{n}\nu_{(p)}^{-1}(R_p(s))^{\frac{p-2}{p-1}}\Bigg)ds, \tag{33}$$

where $R_p(s) := \mathbb{E}\left[\|\theta_s - \theta^*\|_2^{p-2}\,\psi(\|\theta_s - \theta^*\|_2)\right]$.

Taken the above claims as given at the moment, let us now complete the proof of the theorem. Since the process $(\theta_t : t \geq 0)$ converges in $\mathbb{L}^q$ norm for arbitrarily large q, the limit $\lim_{t\to+\infty} R_p(t)$ exists. Since the functions $\tau_{(p)}$ and $\nu_{(p)}$ are convex and strictly increasing, their inverse functions are concave. Moreover, simple calculation leads to

$$\nabla_r\left(\nu_{(p)}^{-1}(r)^{\frac{p-2}{p-1}}\right) = \frac{p-2}{p-1}\frac{\nu_{(p)}^{-1}(r)^{-\frac{1}{p-1}}}{\nu_{(p)}'(\nu_{(p)}^{-1}(r))}. \tag{34}$$

Since $\nu_{(p)}$ is convex and increasing, the numerator is a decreasing positive function of r. Additionally, the denominator is an increasing positive function of r. Therefore, the derivative in equation (34) is a decreasing function of r, and the function $r \mapsto \nu_{(p)}^{-1}(r)^{\frac{p-2}{p-1}}$ is concave. We denote

$$\phi(r) := -r + \varepsilon(n,\delta)\tau_{(p)}^{-1}(r) + \frac{B+(p-1)d}{n}\nu_{(p)}^{-1}(r)^{\frac{p-2}{p-1}}.$$

Then, we have $\phi(0) = 0$ and ϕ is a concave function. Suppose that r_* is the smallest positive solution to the following equation:

$$r = \varepsilon(n,\delta)\tau_{(p)}^{-1}(r) + \frac{B+(p-1)d}{n}\nu_{(p)}^{-1}(r)^{\frac{p-2}{p-1}}.$$

Then, we have $\phi(r) < 0$ for $r > r_*$ and $\phi(r) > 0$ for $r \in (0, r_*)$. By the result of Lemma 1, we have $\lim_{t\to+\infty} R_p(t) \leq r_*$.

Since $\nu_{(p)}$ is a convex and strictly increasing function, by means of Jensen's inequality, we have

$$R_p(t) = \mathbb{E}\left(\|\theta_t - \theta^*\|_2^{p-2}\,\psi(\|\theta_t - \theta^*\|_2)\right) \geq \nu_{(p)}\left(\mathbb{E}\,\|\theta_t - \theta^*\|_2^{p-1}\right).$$

Therefore, by denoting $z_* := \lim_{t\to+\infty}\left(\mathbb{E}\,\|\theta_t - \theta^*\|_2^{p-1}\right)^{\frac{1}{p-1}}$, we have $z_*^{p-1} \leq \nu_{(p)}^{-1}(r_*)$. Hence, we arrive at the following inequality

$$\begin{aligned} z_*^{p-2}\psi(z_*) &\leq \varepsilon(n,\delta)\tau_{(p)}^{-1}\left(\nu_{(p)}(z_*^{p-1})\right) + \frac{B+(p-1)d}{n} z_*^{p-2} \\ &= \varepsilon(n,\delta) z_*^{p-1}\zeta(z_*) + \frac{B+(p-1)d}{n} z_*^{p-2}. \end{aligned}$$

As a consequence, we find that

$$\psi(z_*) \leq \varepsilon(n,\delta)\zeta(z_*)z_* + \frac{B+(p-1)d}{n}.$$

Now, we claim that there exists a unique positive solution to equation (9). Given this claim, replacing p by $(p+1)$ and putting the above results together yields

$$\lim_{t\to+\infty}\left(\mathbb{E}\left(\|\theta_t - \theta^*\|_2^p\right)\right)^{\frac{1}{p}} \leq z_p^*,$$

where z_p^* is the unique positive solution to the following equation:

$$\psi(z) = \varepsilon(n,\delta)\zeta(z)z + \frac{B+pd}{n}.$$

Combining the above inequality with the inequality (6) yields the conclusion of the theorem.

We now return to prove our earlier claims about the behavior of the functions $\nu_{(p)}$, $\tau_{(p)}$, the moment bound (33), and the existence of unique positive solution to equation (9).

Structure of the function $\nu_{(p)}$: Since ψ is a convex and strictly increasing function, by taking the second derivative, we find that

$$\begin{aligned} \nu''_{(p)}(r) &= \nabla_r^2\left(\psi\left(r^{\frac{1}{p-1}}\right) r^{\frac{p-2}{p-1}}\right) \\ &= \frac{1}{p-1} r^{\frac{1}{p-1}-1}\psi''\left(r^{\frac{1}{p-1}}\right) + \frac{1}{p-1} r^{-1}\left(\psi'\left(r^{\frac{1}{p-1}}\right) - r^{-\frac{1}{p-1}}\psi\left(r^{\frac{1}{p-1}}\right)\right) \geq 0 \end{aligned}$$

for all $r > 0$. As a consequence, the function $\nu_{(p)}$ is convex.

Structure of the function $\tau_{(p)}$: This proof exploits Assumption W.3 on the functions ψ and ζ. For any $p \geq 2$, we denote $\zeta_{(p)} : r \to r^{p-1}\zeta(r)$ and $\psi_{(p)} : r \to r^{p-2}\psi(r)$ two strictly increasing functions. Therefore, we can define a function $\tau_{(p)} := \psi_{(p)} \circ \zeta_{(p)}^{-1}$, namely, $\tau_{(p)}(r^{p-1}\zeta(r)) = r^{p-2}\psi(r)$, for any $r > 0$. Following some calculation, we find that

$$\begin{aligned} \nabla_r\left(\tau_{(p)}(r^{p-1}\zeta(r))\right) &= \left[(p-1)r^{p-2}\zeta(r) + r^{p-1}\zeta'(r)\right]\tau'_{(p)}(r^{p-1}\zeta(r)) \\ &= (p-2)r^{p-3}\psi(r) + r^{p-2}\psi'(r). \end{aligned}$$

Setting $z = \zeta_{(p)}(r)$ leads to

$$\nabla_z \tau_{(p)}(z) = \frac{(p-2)\psi(r) + r\psi'(r)}{(p-1)r\zeta(r) + r^2\zeta'(r)}.$$

Taking another derivative of the above term, we find that

$$\nabla_z^2 \tau_{(p)}(z) = \left(\zeta'_{(p)}(r)\right)^{-1} \frac{g(r,p)}{\left((p-1)r\zeta(r) + r^2\zeta'(r)\right)^2},$$

where we denote

$$\begin{aligned} g(r,p) := &\left[(p-1)r\zeta(r) + r^2\zeta'(r)\right] \cdot \left[(p-1)\psi'(r) + r\psi''(r)\right] \\ &- \left[(p-1)\zeta(r) + (p+1)r\zeta'(r) + r^2\zeta''(r)\right] \cdot \left[(p-2)\psi(r) + r\psi'(r)\right]. \end{aligned}$$

According to Assumption W.3, $\tau_{(2)} = \psi_{(2)} \circ \zeta_{(2)}^{-1}$ is a convex function. Therefore, we have $g(r,2) \geq 0$ for any $r > 0$. Simple algebra with first order derivative of function g with respect to parameter p leads to

$$\begin{aligned} \nabla_p \left(g(r,p)\right) =& \zeta(r) \cdot \left[(p-1)r\psi'(r) + r^2\psi''(r) - (p-2)\psi(r) - r\psi'(r)\right] \\ &- r\zeta'(r)\left[(p-2)\psi(r) + r\psi'(r)\right] + r\psi'(r) \cdot \left[(p-1)\zeta(r) + r\zeta'(r)\right] \\ &- \psi(r) \cdot \left[(p-1)\zeta(r) + (p+1)r\zeta'(r) + r^2\zeta''(r)\right] \\ =& 2(p-2)\left[r\psi'(r)\zeta(r) - \psi(r)\zeta(r) - r\zeta'(r)\psi(r)\right] \\ &+ \left[r^2\zeta(r)\psi''(r) + r\psi'(r)\zeta(r) - 3\psi(r)\zeta(r) - r^2\psi(r)\zeta''(r)\right] \geq 0 \end{aligned}$$

for all $r > 0$ where the last inequality is due to Assumption W.3. Therefore, the function g is increasing function in terms of p when $p \geq 2$. It indicates that $g(r,p) \geq g(r,2) \geq 0$ for all $r > 0$. Given that inequality, we have $\frac{d^2}{dz^2}\tau_{(p)}(z) \geq 0$ for any $z \geq 0$, $p \geq 2$, i.e., the function $\tau_{(p)}(z)$ is a convex function for $z = \zeta_{(p)}(r)$.

Proof of claim (33): For any $p \geq 2$, an application of Itô's formula yields the bound $\|\theta_t - \theta^*\|_2^p \leq \sum_{j=1}^5 T_j$, where

$$T_1 := -\frac{p}{2}\int_0^t \langle \theta^* - \theta_s,\, \nabla F(\theta_s)\rangle \|\theta_s - \theta^*\|_2^{p-2}\, ds, \tag{35a}$$

$$T_2 := \frac{p}{2}\int_0^t \langle \theta^* - \theta_s,\, \nabla F(\theta_s) - \nabla F_n(\theta_s)\rangle \|\theta_s - \theta^*\|_2^{p-2}\, ds \tag{35b}$$

$$T_3 := \frac{p}{2n}\int_0^t \langle \theta_s - \theta^*,\, \nabla \log \pi(\theta_s)\rangle \|\theta_s - \theta^*\|_2^{p-2}\, ds \tag{35c}$$

$$T_4 := p\int_0^t \|\theta_s - \theta^*\|_2^{p-2} \langle \theta_s - \theta^*,\, dB_s\rangle \tag{35d}$$

$$T_5 := \frac{p(p-1)d}{2n}\int_0^t \|\theta_s - \theta^*\|_2^{p-2}\, ds. \tag{35e}$$

We now upper bound the terms $\{T_j\}_{j=1}^5$ in terms of functionals of the quantity R_p. From the weak convexity of F guaranteed by Assumption W.1, we have

$$\mathbb{E}\left[T_1\right] = -\frac{p}{2}\mathbb{E}\left[\int_0^t \langle \theta^* - \theta_s,\, \nabla F(\theta_s)\rangle \|\theta_s - \theta^*\|_2^{p-2}\, ds\right] \leq -\frac{p}{2}\int_0^t R_p(s) ds. \tag{36a}$$

Based on Assumption W.2, we find that

$$\begin{aligned}\mathbb{E}\left[T_2\right] = \frac{p}{2}\mathbb{E}\left[\int_0^t \langle \theta^* - \theta_s,\, \nabla F(\theta_s) - \nabla F_n(\theta_s)\rangle \left\|\theta_s - \theta^*\right\|_2^{p-2} ds\right] \\ \leq \frac{p}{2}\varepsilon(n,\delta)\int_0^t \mathbb{E}\left[\left\|\theta_s - \theta^*\right\|_2^{p-1} \zeta(\left\|\theta_s - \theta^*\right\|_2)\right] ds.\end{aligned}$$

Since the function $\tau_{(p)}$ is convex, invoking Jensen's inequality, we obtain the following inequalities:

$$\begin{aligned}\int_0^t \mathbb{E}\left[\left\|\theta_s - \theta^*\right\|_2^{p-1} \zeta\left(\left\|\theta_s - \theta^*\right\|_2\right)\right] ds &\leq \int_0^t \tau_{(p)}^{-1}\mathbb{E}\left[\tau_{(p)}\left(\left\|\theta_s - \theta^*\right\|_2^{p-1} \zeta(\left\|\theta_s - \theta^*\right\|_2)\right)\right] ds \\ &= \int_0^t \tau_{(p)}^{-1}\left(R_p(s)\right) ds.\end{aligned}$$

In light of the above inequalities, we have

$$\mathbb{E}\left[T_2\right] \leq \frac{p}{2}\varepsilon(n,\delta)\int_0^t \tau_{(p)}^{-1}\left(R_p(s)\right) ds. \tag{36b}$$

Moving to T_3 in equation (35c), given Assumptions A and B about the smoothness of prior distribution π, its expectation is bounded as

$$\begin{aligned}\mathbb{E}\left[T_3\right] &= \frac{p}{2n}\mathbb{E}\left[\int_0^t \langle \theta_s - \theta^*,\, \nabla \log \pi(\theta_s)\rangle \left\|\theta_s - \theta^*\right\|_2^{p-2} ds\right] \\ &\leq \frac{pB}{2n}\int_0^t \mathbb{E}\left[\left\|\theta_s - \theta^*\right\|_2^{p-2}\right] ds \end{aligned} \tag{36c}$$

Since $\nu_{(p)}$ is a strictly increasing and convex function on $[0,+\infty)$, the function $\nu_{(p)}^{-1}$ is a concave function on $[0,+\infty)$. Invoking Jensen's inequality leads to the following inequality:

$$\begin{aligned}\mathbb{E}\left[\int_0^t \left\|\theta_s - \theta^*\right\|_2^{p-2} ds\right] \leq \int_0^t \left(\mathbb{E}\left[\left\|\theta_s - \theta^*\right\|_2^{p-1}\right]\right)^{\frac{p-2}{p-1}} ds &\leq \int_0^t \nu_{(p)}^{-1}\left(\mathbb{E}\nu_{(p)}\left(\left\|\theta_s - \theta^*\right\|_2^{p-1}\right)\right)^{\frac{p-2}{p-1}} ds \\ &\leq \int_0^t \nu_{(p)}^{-1}\left(R_p(s)\right)^{\frac{p-2}{p-1}} ds.\end{aligned} \tag{36d}$$

Combining the inequalities (36c) and (36d), we have

$$\mathbb{E}\left[T_3\right] \leq \frac{pB}{2n}\int_0^t \nu_{(p)}^{-1}\left(R_p(s)\right)^{\frac{p-2}{p-1}} ds. \tag{36e}$$

Moving to the fourth term T_4 from equation (35d), we have

$$\mathbb{E}\left[T_4\right] = \mathbb{E}\left[\int_0^t \left\|\theta_s - \theta^*\right\|_2^{p-2} \langle \theta_s - \theta^*,\, dB_s\rangle\right] = 0, \tag{36f}$$

where we have used the martingale structure. Furthermore, in light of moment bound in equation (36d), we have

$$\mathbb{E}\left[T_5\right] \leq \frac{p(p-1)d}{2n}\int_0^t \nu_{(p)}^{-1}\left(R_p(s)\right)^{\frac{p-2}{p-1}} ds. \tag{36g}$$

Collecting the bounds on the expectations of $\{T_j\}_{j=1}^5$ from equations (36a), (36b), (36e), (36f), and (36g), respectively, yields the claim (33).

Unique positive solution to equation (9): We now establish that equation (9) has a unique positive solution under the stated assumptions. Define the function

$$\vartheta(z) := \psi(z) - \left(\varepsilon(n,\delta)\zeta(z)z + \frac{B + d\log(1/\delta)}{n}\right).$$

Since $\psi(0) = 0$, we have $\vartheta(0) < 0$. On the other hand, based on Assumption W.4, $\liminf_{z\to+\infty} \vartheta(z) > 0$. Therefore, there exists a positive solution to the equation $\vartheta(z) = 0$.

Recall that $\xi : \mathbb{R}_+ \to \mathbb{R}$ is an inverse function of the strictly increasing function $z \mapsto z\zeta(z)$. Therefore, we can write the function ϑ as follows:

$$\vartheta(z) = \widetilde{\vartheta}(r) := \psi(\xi(r)) - \varepsilon(n,\delta)r - \frac{B + d\log(1/\delta)}{n},$$

where $r = z \cdot \zeta(z)$. Given the convexity of function $r \mapsto \psi(\xi(r))$ guaranteed by Assumption W.3, the functions $\widetilde{\vartheta}$ and ϑ are convex. Putting the above results together, there exists a unique positive solution to equation (9).

6 Discussion

In this paper, we described an approach for analyzing the posterior contraction rates of parameters based on the diffusion processes. Our theory depends on two important features: the convex-analytic structure of the population log-likelihood function F and stochastic perturbation bounds between the gradient of F and the gradient of its sample counterpart F_n. For log-likelihoods that are strongly concave around the true parameter θ^*, we established posterior convergence rates for estimating parameters of the order $(d/n)^{1/2}$ along with other model parameters, valid under sufficiently smooth conditions of prior distribution π and mild conditions on the perturbation error between ∇F_n and ∇F. On the other hand, when the population log-likelihood function is weakly concave, our analysis shows that convergence rates are more delicate: they depend on an interaction between the degree of weak convexity, and the stochastic error bounds. In this setting, proved that the posterior convergence rate of parameter is upper bounded by the unique positive solution of a non-linear equation determined by the previous interplay. As an illustration of our general theory, we derived posterior convergence rates for three concrete examples: Bayesian logistic regression models, Bayesian single index models, and overspecified Bayesian location Gaussian mixture models.

Let us now discuss a few directions that arise naturally from our work. First, the current results are not directly applicable to establish the asymptotic convergence of posterior distribution of parameter under locally weakly concave settings of F around the true parameter θ^*. More precisely, when F is locally strongly concave around θ^*, it is well-known from Berstein-von Mises theorem that the posterior distribution of parameter converges to a multivariate normal distribution centered at the maximum likelihood estimation (MLE) with the covariance matrix is given by $1/\left(nI(\theta^*)\right)$ (e.g., see the book [32]), where $I(\theta^*)$ denotes the Fisher information matrix at θ^*. Under the weak concavity setting of F, the Fisher information matrix $I(\theta^*)$ is degenerate. Therefore, the posterior distribution of parameter can no longer be approximated by a multivariate Gaussian distribution in this setting.

Second, it is desirable to understand the posterior convergence rate of parameter under the non-convex settings of population log-likelihood function F, which arise in various statistical models such as Latent Dirichlet Allocation [3]. Under these settings, without a careful analysis of the multi-modal structures of the population and sample log-likelihood functions, the diffusion approach may lead to exponential dependence on dimension d and sub-optimal dependence on other model parameters.

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A Proofs of corollaries

In this appendix, we collect the proofs of several corollaries stated in Section 4. To summarize, we make use of Theorem 2 to establish the posterior contraction rates of parameters in all three examples. The crux of the proofs of these corollaries involves a verification of Assumptions W.1, W.2, and W.3 to invoke the respective theorem. We first present the proof of Corollary 2 in Appendix A.1. Then, we move to the proofs of Corollary 3 and Corollary 4 in Appendices A.2 and A.3 respectively. Note that the values of universal constants may change from line-to-line.

A.1 Proof of Corollary 2

We begin by verifying claim (17a) about the structure of the negative population log-likelihood function F^R and claim (17b) about the uniform perturbation error between ∇F^R and ∇F_n^R.

A.1.1 Proof of claim (17a)

Following some algebra, we find that

$$
\begin{aligned}
-F^R(\theta) &= \mathbb{E}\left[-Y \log\left(1 + e^{-\langle X,\,\theta\rangle}\right) - (1-Y)\log\left(1 + e^{\langle X,\,\theta\rangle}\right)\right] \\
&= -\mathbb{E}\left[\frac{1}{1+e^{-\langle X,\,\theta^*\rangle}} \log\left(1 + e^{-\langle X,\,\theta\rangle}\right) + \frac{1}{1+e^{\langle X,\,\theta^*\rangle}} \log\left(1 + e^{\langle X,\,\theta\rangle}\right)\right],
\end{aligned}
$$

where the above expectations are taken with respect to $X \sim \mathcal{N}(0, \sigma^2 I_d)$ and $Y|X$ following probability distribution generated from logistic model (14). By taking derivative of $F^R(\theta)$, we obtain that

$$
\langle \nabla F^R(\theta),\, \theta^* - \theta\rangle = \mathbb{E}\left[\left(\frac{1+e^{\langle X,\,\theta\rangle}}{1+e^{\langle X,\,\theta^*\rangle}} - \frac{1+e^{-\langle X,\,\theta\rangle}}{1+e^{-\langle X,\,\theta^*\rangle}}\right)\frac{e^{-\langle X,\,\theta\rangle}}{(1+e^{-\langle X,\,\theta\rangle})^2}\langle X,\, \theta - \theta^*\rangle\right].
$$

By the mean value theorem, there exists ξ between 0 and $\langle X,\, \theta - \theta^*\rangle$ such that

$$
\frac{1+e^{\langle X,\,\theta\rangle}}{1+e^{\langle X,\,\theta^*\rangle}} - \frac{1+e^{-\langle X,\,\theta\rangle}}{1+e^{-\langle X,\,\theta^*\rangle}} = \langle X,\, \theta - \theta^*\rangle\left(\frac{e^{\langle X,\,\theta^*\rangle + \xi}}{1+e^{\langle X,\,\theta^*\rangle}} + \frac{e^{-\langle X,\,\theta^*\rangle - \xi}}{1+e^{-\langle X,\,\theta^*\rangle}}\right).
$$

In light of the above equality, we arrive at the following inequalities:

$$\begin{aligned}
\langle \nabla F^R(\theta),\, \theta^* - \theta \rangle &\geq \mathbb{E}\Bigg[\inf_{|\xi| \in [0, |\langle X,\, \theta - \theta^* \rangle|]} \left(\frac{e^{\langle X,\, \theta^* \rangle + \xi}}{1 + e^{\langle X,\, \theta^* \rangle}} + \frac{e^{-\langle X,\, \theta^* \rangle - \xi}}{1 + e^{-\langle X,\, \theta^* \rangle}} \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \frac{e^{-\langle X,\, \theta \rangle}}{(1 + e^{-\langle X,\, \theta \rangle})^2} |\langle X,\, \theta - \theta^* \rangle|^2 \Bigg] \\
&\geq \mathbb{E}\left[\frac{1}{2} e^{-|\langle X,\, \theta - \theta^* \rangle|} \frac{e^{-\langle X,\, \theta \rangle}}{(1 + e^{-\langle X,\, \theta \rangle})^2} |\langle X,\, \theta - \theta^* \rangle|^2 \right] \\
&\geq \frac{1}{8} \mathbb{E}\left[e^{-|\langle X,\, \theta - \theta^* \rangle| - |\langle X,\, \theta \rangle|} |\langle X,\, \theta - \theta^* \rangle|^2 \right] \\
&\geq \frac{1}{8e^4} \mathbb{E}\left[\mathbf{1}_{\{|\langle X,\, \theta \rangle| \leq 2,\ |\langle X,\, \theta - \theta^* \rangle| \leq 2\}} |\langle X,\, \theta - \theta^* \rangle|^2 \right].
\end{aligned}$$

Since $X \sim \mathcal{N}(0, I_d)$, we have $\begin{bmatrix} \langle X,\, \theta \rangle \\ \langle X,\, \theta - \theta^* \rangle \end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} \|\theta\|_2^2 & \langle \theta,\, \theta - \theta^* \rangle \\ \langle \theta,\, \theta - \theta^* \rangle & \|\theta - \theta^*\|_2^2 \end{bmatrix}\right)$. Given that result, direct calculation leads to

$$\mathbb{E}\left(\mathbf{1}_{\{|\langle X,\, \theta \rangle| \leq 2, |\langle X,\, \theta - \theta^* \rangle| \leq 2\}} |\langle X,\, \theta - \theta^* \rangle|^2 \right) \geq \frac{c}{(1 + \|\theta\|_2)(1 + \|\theta - \theta^*\|_2)} \|\theta - \theta^*\|_2^2,$$

for a universal constant $c > 0$. Collecting the above results, for all θ such that $\|\theta - \theta^*\|_2 \leq 1$, we achieve that

$$\begin{aligned}
\langle \nabla F^R(\theta),\, \theta^* - \theta \rangle &\geq \frac{c}{(1 + \|\theta\|_2)(1 + \|\theta - \theta^*\|_2)} \|\theta - \theta^*\|_2^2 \\
&\geq c \frac{1}{1 + \|\theta^*\|_2} \|\theta - \theta^*\|_2^2.
\end{aligned}$$

For θ with $\|\theta - \theta^*\|_2 > 1$, let $\tilde{\theta} = \theta^* + \frac{\theta - \theta^*}{\|\theta - \theta^*\|_2}$. Then, we find that

$$\langle \nabla F^R(\theta),\, \theta^* - \theta \rangle \geq \langle \nabla F^R(\tilde{\theta}),\, \theta^* - \theta \rangle \geq \frac{c}{2(1 + \|\theta^*\|_2)} \|\theta - \theta^*\|_2,$$

which yields the claim (17a).

A.1.2 Proof of the bound (17b)

In this appendix, we prove the uniform bound (17b) between the empirical and population likelihood gradients. It suffices to establish the following stronger result:

$$Z \;:=\; \sup_{\theta \in \mathbb{R}^d} \left\| \nabla F_n^R(\theta) - \nabla F^R(\theta) \right\|_2 \leq c \left\{ \sqrt{\frac{d}{n}} + \sqrt{\frac{\log(1/\delta)}{n}} + \frac{\log(1/\delta)}{n} \right\}, \tag{37}$$

with probability at least $1 - \delta$ for any $\frac{n}{\log n} \geq c_0 d \log(1/\delta)$ where c_0 is a universal constant.

We begin by writing Z as the supremum of a stochastic process. Let $\mathbb{S}^{d-1}$ denote the Euclidean sphere in $\mathbb{R}^d$, and define the stochastic process

$$Z_{u,\theta} := \left| \frac{1}{n} \sum_{i=1}^n f_{u,\theta}(X_i, Y_i) - \mathbb{E}[f_{u,\theta}(X, Y)] \right|, \quad \text{where } f_{u,\theta}(x, y) = \frac{y \langle x,\, u \rangle e^{y \langle x,\, \theta \rangle}}{1 + e^{y \langle x,\, \theta \rangle}},$$

indexed by vectors $u \in \mathbb{S}^{d-1}$ and $\theta \in \mathbb{B}(\theta^*; r)$. The outer expectation in the above display is taken with respect to (X, Y) from the logistic model (14). Observe that $Z = \sup_{u \in \mathbb{S}^{d-1}} \sup_{\theta \in \mathbb{R}^d} Z_{u,\theta}$. Let $\{u^1, \ldots, u^N\}$ be a 1/8-covering of $\mathbb{S}^{d-1}$ in the Euclidean norm; there exists such a set with $N \le 17^d$ elements. By a standard discretization argument (see Chapter 6, [35]), we have

$$Z \le 2 \max_{j=1,\ldots,N} \sup_{\theta \in \mathbb{R}^d} Z_{u^j,\theta}.$$

Accordingly, the remainder of our argument focuses on bounding the random variable $V := \sup_{\theta \in \mathbb{R}^d} Z_{u,\theta}$, where the vector $u \in \mathbb{S}^{d-1}$ should be understood as arbitrary but fixed.

Define the symmetrized random variable

$$V' := \sup_{\theta \in \mathbb{R}^d} \left| \frac{1}{n} \sum_{i=1}^n \varepsilon_i \langle X_i,\, u \rangle \varphi_\theta(X_i, Y_i) \right|, \quad \text{where } \varphi_\theta(x, y) = y \frac{e^{y\langle x,\, \theta \rangle}}{1 + e^{y \langle x,\, \theta \rangle}},$$

and $\{\varepsilon_i\}_{i=1}^n$ is an i.i.d. sequence of Rademacher variables. By a symmetrization inequality for probabilities [33], we have

$$\mathbb{P}\left[V \ge t\right] \le c_1 \mathbb{P}\left[V' \ge c_2 t\right],$$

where c_1 and c_2 are two positive universal constants. We now analyze the Rademacher process that defines V' conditionally on $\{(X_i, Y_i)\}_{i=1}^n$. We first use a functional Bernstein inequality to control the deviations above its expectation. For a parameter $b > 0$ to be chosen, define the event

$$\mathcal{E}_b := \left\{ \frac{1}{n} \sum_{i=1}^n \langle X_i,\, u \rangle^2 \le 2 \text{ and } |\langle X_i,\, u\rangle| \le b \text{ for all } i = 1, \ldots, n \right\}.$$

Conditioned on $\mathcal{E}_b$, we have $|\langle X_i,\, u\rangle \varphi_\theta(X_i, Y_i)| \le b$ for all $i = 1, \ldots, n$. Moreover, we have

$$\Sigma^2 := \sup_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n (\langle X_i,\, u\rangle)^2 \varphi_\theta^2(X_i, Y_i) \le 2.$$

Consequently, by Talagrand's theorem on empirical processes (Theorem 3.27 in [35]), we find that

$$\mathbb{P}\left[V' \ge \mathbb{E}[V'] + s \mid \mathcal{E}_b\right] \le 2 \exp\left(-\frac{ns^2}{16e + 4bs}\right) \qquad \text{for all } s > 0.$$

We now bound the probability $\mathbb{P}[\mathcal{E}_b^c]$ for a suitable choice of b. By standard χ^2-tail bounds (Example 2.11, [35]), we have $\mathbb{P}[\frac{1}{n}\sum_{i=1}^n \langle X_i,\, u\rangle^2 \ge 2] \le e^{-n/8}$. By concentration of Gaussian maxima (Example 2.29, [35]), we have

$$\mathbb{P}\left[\max_{i=1,\ldots,n} |\langle X_i,\, u\rangle| \ge 4\sqrt{\log n} + t\right] \le e^{-nt^2/2}.$$

Setting $t = \sqrt{\frac{n}{d}}$ and using the assumption that $\frac{n}{\log n} \ge c_0 d$, we find that

$$\mathbb{P}\Big[\max_{i=1,\ldots,n} |\langle X_i,\, u\rangle| \ge b\Big] \le e^{-c_2 \frac{n^2}{d}} \qquad \text{where } b := c_1 \sqrt{\tfrac{n}{d}}.$$

Consequently, we have

$$\mathbb{P}\left[V' \geq \mathbb{E}[V'] + s\right] \leq \exp\left(-\frac{ns^2}{16e + c_3\sqrt{\frac{n}{d}}s}\right) + e^{-c_2\frac{n^2}{d}} + e^{-n/8}$$
$$\leq e^{-\frac{ns^2}{32e}} + e^{-\frac{s\sqrt{nd}}{2c_3}} + e^{-c_2\frac{n^2}{d}} + e^{-n/8}. \tag{38}$$

We now bound the expectation of V', first over the Rademacher variables. Consider the following function class:

$$\mathcal{G} := \left\{g_\theta : (x, y) \mapsto \langle x,\, u\rangle \varphi_\theta(x, y) \;\mid\; \theta \in \mathbb{R}^d\right\}.$$

It is clear that the function class $\mathcal{G}$ has the envelope function $\bar{G}(x) := |\langle x,\, u\rangle|$. We claim that the L_2-covering number of $\mathcal{G}$ can be bounded as

$$\bar{N}(t) := \sup_Q \left|\mathcal{N}\left(\mathcal{G}, \|\cdot\|_{L^2(Q)}, t\left\|\bar{G}\right\|_{L^2(Q)}\right)\right| \leq \left(\frac{1}{t}\right)^{c(d+1)} \qquad \text{for all } t > 0, \tag{39}$$

where $c > 0$ is a universal constant.

Let us take the claim (39) as given for the moment, and use it to bound the expectation of V', first over the Rademacher variables. Define the empirical expectation $\mathbb{P}_n(\bar{G}^2) := \frac{1}{n}\sum_{i=1}^n \langle X_i,\, u\rangle^2$. Invoking Dudley's entropy integral bound (e.g., Theorem 5.22, [35]), we find that there are universal constants C, C' such that

$$\mathbb{E}_\varepsilon[V'] = \mathbb{E}_\varepsilon\left[\sup_{g\in\mathcal{G}}\left|\frac{1}{n}\sum_{i=1}^n \varepsilon_i g(X_i, Y_i)\right|\right] \leq C\sqrt{\frac{\mathbb{P}_n(\bar{G}^2)}{n}}\int_0^1 \sqrt{1 + \log \bar{N}(t)}dt$$
$$\leq C'\sqrt{\mathbb{P}_n(\bar{G}^2)}\sqrt{\frac{d}{n}}.$$

Up to this point, we have been conditioning on the observations $\{X_i\}_{i=1}^n$. Taking expectations over them as well yields

$$\mathbb{E}_{\varepsilon, X_1^n}[V'] \leq C'\sqrt{\frac{d}{n}} \cdot \mathbb{E}_{X_1^n}\left[\sqrt{\mathbb{P}_n(\bar{G}^2)}\right] \overset{(i)}{\leq} C'\sqrt{\frac{d}{n}} \cdot \sqrt{\mathbb{E}_{X_1^n}\left[\mathbb{P}_n(\bar{G}^2)\right]} \overset{(ii)}{=} C'\sqrt{\frac{d}{n}}, \tag{40}$$

where step (i) follows from Jensen's inequality; and step (ii) uses the fact that $\mathbb{E}_{X_1^n}[\mathbb{P}_n(\bar{G}^2)] = 1$. Putting together the bounds (38) and (40) yields

$$\mathbb{P}\left[V' \geq C'\sqrt{\frac{d}{n}} + s\right] \leq e^{-\frac{ns^2}{32e}} + e^{-\frac{s\sqrt{nd}}{2c_3}} + e^{-c_2\frac{n^2}{d}} + e^{-n/8}.$$

This probability bound holds for each $u \in \mathbb{S}^{d-1}$. By taking the union bound over the 1/8-covering set $\{u^1, \ldots, u^N\}$ of $\mathbb{S}^{d-1}$ where $N \leq 17^d$ and choosing $s = c'\left(\sqrt{\frac{d}{n}} + \sqrt{\frac{\log(1/\delta)}{n}} + \frac{\log(1/\delta)}{n}\right)$ for a sufficiently large constant $c' > 0$, we obtain the claim (37).

Proof of claim (39): We consider a fixed sequence $(x_i, y_i, t_i)_{i=1}^m$ where $y_i \in \{-1, 1\}$, $x_i \in \mathbb{R}^d$ and $t_i \in \mathbb{R}$ for $i \in [m]$. Now, we suppose that for any binary sequence $(z_i)_{i=1}^m \in \{0,1\}^m$, there exists $\theta \in \mathbb{R}^d$ such that

$$z_i = \mathbb{I}\left[\langle X_i,\, u\rangle \varphi_\theta(X_i, Y_i) \geq t_i\right] \qquad \text{for all } i \in [m].$$

Following some algebra, we find that

$$y_i x_i^T \theta - \log \frac{Y_i t_i}{\langle X_i,\, u\rangle - Y_i t_i} \begin{cases} \geq 0 & z_i = 1 \\ < 0 & z_i = 0 \end{cases}.$$

Consequently, the set $\{[y_i x_i, \log(Y_i t_i/(\langle X_i,\, u\rangle - Y_i t_i))]\}_{i=1}^m$ of $(d+1)$-dimensional points can be shattered by linear separators. Therefore, we have $m \leq d+2$, which leads to the VC subgraph dimension of $\mathcal{G}$ to be at most $d+2$ (e.g., see the book [34]). As a consequence, we obtain the conclusion of the claim (39).

A.2 Proof of Corollary 3

The claim (24a) of weak convexity for the negative population log-likelihood function F^I is straightforward. Therefore, we only need to establish the claim (24b) about the uniform perturbation bound between ∇F^I and ∇F_n^I.

A.2.1 Bounding the difference $\nabla F^I - \nabla F_n^I$

It is convenient to introduce the shorthand

$$p_\theta(x, y) = \left(y - \left(x^\top \theta\right)^p\right)^2 / 2 \qquad \text{for all } (x, y) \in \mathbb{R}^{d+1}.$$

We then compute the gradient

$$\nabla \log p_\theta(x, y) = p\left(y - \left(x^\top \theta\right)^p\right)\left(x^\top \theta\right)^{p-1} x.$$

Fix an arbitrary $r > 0$, by applying the triangle inequality, we find that

$$\begin{aligned} \sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \nabla F_n^I(\theta) - \nabla F^I(\theta) \right\|_2 &= \sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \frac{1}{n} \sum_{i=1}^n \nabla \log p_\theta(X_i, Y_i) - \mathbb{E}_{(X,Y)}\left[\nabla \log p_\theta(X, Y)\right] \right\|_2 \\ &\leq p\left\{J_1 + J_2\right\}, \end{aligned}$$

where we define

$$J_1 := p \sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \frac{1}{n} \sum_{i=1}^n Y_i X_i \left(X_i^\top \theta\right)^{p-1} \right\|_2, \quad \text{and} \tag{41a}$$

$$J_2 := p \sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \frac{1}{n} \sum_{i=1}^n X_i \left(X_i^\top \theta\right)^{2p-1} - \mathbb{E}_X\left[X\left(X^\top \theta\right)^{2p-1}\right] \right\|_2. \tag{41b}$$

We claim that there is a universal constant c such that for any $\delta \in (0,1)$, the quantities J_1 and J_2 can be bounded as

$$J_1 \leq c\, r^{p-1} \left(\sqrt{\frac{d+\log\frac{1}{\delta}}{n}} + \frac{1}{n^{3/2}} \left(d + \log\frac{n}{\delta}\right)^{p+1} \right), \quad \text{and} \tag{42a}$$

$$J_2 \leq c\, r^{2p-1} \left(\sqrt{\frac{d+\log\frac{1}{\delta}}{n}} + \frac{1}{n^{3/2}} \left(d + \log\frac{n}{\delta}\right)^{2p+1} \right), \tag{42b}$$

with probability at least $1-\delta$.

Assume that the above claims are given at the moment. We proceed to finish the proof of the uniform perturbation bound between ∇F_n^I and ∇F^I in (24b). In fact, plugging the concentration bounds (42a) and (42b) into (41), we obtain that

$$\sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \nabla F_n^I(\theta) - \nabla F^I(\theta) \right\|_2 \leq c\left(r^{p-1} + r^{2p-1}\right) \sqrt{\frac{d + \log(1/\delta)}{n}}$$
$$+ \frac{r^{p-1}(d+\log(1/\delta)+\log n)^{p+1} + r^{2p-1}(d+\log(1/\delta)+\log n)^{2p+1}}{n^{\frac{3}{2}}},$$

for any $r > 0$ with probability at least $1 - 2\delta$ where c is a universal constant. When $n \geq c' \left(d + \log(d/\delta)\right)^{2p}$ for some universal constant c', it is clear that the the second term is dominated by the first term in the RHS of the above inequality. As a consequence, we have proved the claim (24b).

Proof of claim (42a): Following some algebra, we find that

$$\sup_{r>0} \frac{\sup_{\theta\in\mathbb{B}(\theta^*,r)} \left\| \frac{1}{n}\sum_{i=1}^n Y_i X_i \left(X_i^\top \theta\right)^{p-1} \right\|_2}{r^{p-1}} \leq \sup_{r>0} \sup_{\theta\in\mathbb{B}(\theta^*,r)} \left\| \frac{1}{n}\sum_{i=1}^n Y_i X_i \left(X_i^\top \frac{\theta}{\|\theta\|_2}\right)^{p-1} \right\|_2$$
$$= \underbrace{\sup_{\theta\in\mathbb{S}^{d-1}} \left\| \frac{1}{n}\sum_{i=1}^n Y_i X_i \left(X_i^\top \theta\right)^{p-1} \right\|_2}_{=:Z}. \tag{43}$$

Thus, in order to establish the claim (42a), it suffices to show that there is a universal constant c such that

$$\mathbb{P}\left(Z \leq c\sqrt{\frac{d+\log(1/\delta)}{n}} + \frac{1}{n^{3/2}}\left(d + \log\frac{n}{\delta}\right)^{p+1} \right) \geq 1-\delta. \tag{44}$$

By the variational definition of the Euclidean norm, we have

$$Z = \sup_{\theta\in\mathbb{S}^{d-1}} \left\| \frac{1}{n}\sum_{i=1}^n Y_i X_i \left(X_i^\top \theta\right)^{p-1} \right\|_2 = \sup_{u\in\mathbb{S}^{d-1}} \underbrace{\sup_{\theta\in\mathbb{S}^{d-1}} \left| \frac{1}{n}\sum_{i=1}^n Y_i X_i^\top u \left(X_i^\top \theta\right)^{p-1} \right|}_{:=Z_u}.$$

Using a discretization argument as in Appendix A.1.2, we find that

$$Z \leq 2 \sup_{u\in\mathcal{N}\left(\frac{1}{8}, \mathbb{S}^{d-1}, \|\cdot\|_2\right)} Z_u,$$

where $\mathcal{N}\left(\frac{1}{8}, \mathbb{S}^{d-1}, \|.\|_2\right)$ is the $\frac{1}{8}$-covering of $\mathbb{S}^{d-1}$ under $\|.\|_2$ norm. Therefore, it is sufficient to bound Z_u for any fixed $u \in \mathcal{N}\left(\frac{1}{8}, \mathbb{S}^{d-1}, \|.\|_2\right)$.

For any even integer $q \geq 2$, a symmetrization argument (e.g., Theorem 4.10, [35]) yields

$$\mathbb{E}\left(\sup_{\theta \in \mathbb{S}^{d-1}} \left|\frac{1}{n}\sum_{i=1}^{n} Y_i X_i^\top u \left(X_i^\top \theta\right)^{p-1}\right|\right)^q \leq \mathbb{E}\left(\sup_{\theta \in \mathbb{S}^{d-1}} \left|\frac{2}{n}\sum_{i=1}^{n} \varepsilon_i Y_i X_i^\top u \left(X_i^\top \theta\right)^{p-1}\right|\right)^q$$

where $\{\varepsilon_i\}_{i=1}^n$ is an i.i.d. sequence of Rademacher variables. In order to facilitate the proof argument, for any $t > 0$, we introduce the shorthand $\mathcal{N}(t) := \mathcal{N}\left(t, \mathbb{S}^{d-1}, \|.\|_2\right) = \{\theta_1, \ldots, \theta_{\bar{N}(t)}\}$ where $\bar{N}(t) = \left|\mathcal{N}\left(t, \mathbb{S}^{d-1}, \|.\|_2\right)\right|$. For any compact set $\Omega \subseteq \mathbb{R}^d$, we define the following random variable:

$$\mathcal{R}(\Omega) := \sup_{\theta \in \Omega, p' \in [1,p]} \left|\frac{2}{n}\sum_{i=1}^{n} \varepsilon_i Y_i X_i^\top u \left(X_i^\top \theta\right)^{p'-1}\right|.$$

By the definition of t-covering, we obtain that

$$\begin{aligned}
\mathcal{R}(\mathbb{S}^{d-1}) &= \sup_{\theta \in \mathbb{S}^{d-1}, p' \in [1,p]} \left|\frac{2}{n}\sum_{i=1}^{n} \varepsilon_i Y_i X_i^\top u \left(X_i^\top \theta\right)^{p'-1}\right| \\
&\leq \sup_{\theta_k \in \mathcal{N}(t), \|\eta\|_2 \leq t, p' \in [1,p]} \left|\frac{2}{n}\sum_{i=1}^{n} \varepsilon_i Y_i X_i^\top u \left(X_i^\top (\theta_k + \eta)\right)^{p'-1}\right| \\
&\leq \sup_{\theta_k \in \mathcal{N}(t), p' \in [1,p]} \left|\frac{4}{n}\sum_{i=1}^{n} \varepsilon_i Y_i X_i^\top u \left(X_i^\top \theta\right)^{p'-1}\right| \\
&\qquad\qquad + \max_{p' \in [1,p]} \sum_{b=1}^{p'-1} \binom{p'-1}{b} \cdot \sup_{\|\eta\|_2 \leq t} \left|\frac{4}{n}\sum_{i=1}^{n} \varepsilon_i Y_i \langle X_i,\, u\rangle \langle X_i,\, \eta\rangle^b\right| \\
&\leq \mathcal{R}(\mathcal{N}(t)) + 2^{p+1} t \cdot \mathcal{R}(\mathbb{S}^{d-1}).
\end{aligned} \tag{45}$$

By choosing $t = 2^{-(p+2)}$, the above inequality leads to $\mathcal{R}(\mathbb{S}^{d-1}) \leq 2\mathcal{R}(\mathcal{N}(2^{-(p+2)}))$.

In order to obtain a high-probability upper bound on $\mathcal{R}(\mathcal{N}(2^{-(p+2)}))$, we bound its moments. By the union bound, for any $q \geq 1$, we have

$$\mathbb{E}\left[\mathcal{R}^q\left(\mathcal{N}(2^{-(p+2)})\right)\right] \leq p \cdot \left|\mathcal{N}(2^{-(p+2)})\right| \cdot \sup_{\theta \in \mathbb{S}^{d-1}, p' \in [1,p]} \underbrace{\mathbb{E}\left[\left(\left|\frac{4}{n}\sum_{i=1}^{n} \varepsilon_i Y_i X_i^\top u \left(X_i^\top \theta\right)^{p'-1}\right|\right)^q\right]}_{:= T_1(\theta, p')}.$$

In order to upper bound $T_1(\theta, p')$, we apply Khintchine's inequality [4]; it guarantees that there is a universal constant C such that

$$T_1(\theta, p') \leq \mathbb{E}\left[\left(\frac{Cq}{n^2}\sum_{i=1}^{n} Y_i^2 (X_i^\top u)^2 (X_i^\top \theta)^{2(p'-1)}\right)^{\frac{q}{2}}\right], \tag{46a}$$

for any $p' \in [1, p]$. In order to further upper bound the right hand side, we define the function $g_{\theta,u}(x, y) := y^2 (x^\top u)^2 (x^\top \theta)^{2(p'-1)}$. For any $i \in [n]$, we can verify that

$$\begin{aligned}
\mathbb{E}\left[g_{\theta,u}(X_i, Y_i)\right] =& \mathbb{E}\left[Y_i^2 \cdot \mathbb{E}\left((X_i^\top u)^2 (X_i^\top \theta)^{2(p-1)}\right)\right] \leq (2p')^{p'}, \\
\mathbb{E}\left[g_{\theta,u}(X_i, Y_i)^q\right] =& \mathbb{E}\left[Y_i^{2q} \cdot \mathbb{E}\left((X_i^\top u)^{2q} (X_i^\top \theta)^{2(p'-1)q}\right)\right] \leq (2q)^q (2p'q)^{p'q}.
\end{aligned}$$

Given the above bounds, invoking the result of Lemma 2 leads to the following probability bound

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} g_{\theta,u}(X_i,Y_i) - \mathbb{E}_{(X,Y)}\left[g_{\theta,u}(X,Y)\right]\right| > (8p')^{p'}\sqrt{\frac{\log 4/\delta}{n}} + \frac{1}{n}\left(2p'\log\frac{n}{\delta}\right)^{p+1}\right) \leq \delta,$$

for all $\delta > 0$ where the outer expectation in the above display is taken with respect to (X, Y) such that $X \sim \mathcal{N}(0, I_d)$ and $Y|X = x \sim \mathcal{N}((x^\top\theta^*)^p, 1)$. Putting the previous bounds together, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} g_{\theta,u}(X_i,Y_i)\right)^{q/2}\right] \\
&\leq 2^{q/2}\left(\mathbb{E}_{(X,Y)}\left[g_{\theta,u}(X,Y)\right]\right)^{q/2} + 2^{q/2}\mathbb{E}\left[\left|\frac{1}{n}\sum_{i=1}^{n} g_{\theta,u}(X_i,Y_i) - \mathbb{E}_{(X,Y)}\left[g_{\theta,u}(X,Y)\right]\right|^{q/2}\right] \\
&\leq (4p')^{pq} + q\int_0^{+\infty} \lambda^{q-1}\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} g_{\theta,u}(X_i,Y_i) - \mathbb{E}_{(X,Y)}\left[g_{\theta,u}(X,Y)\right]\right| > \lambda\right) d\lambda \\
&\leq (4p')^{p'q} + q\int_0^{1} (p'+1)\left((8p')^{p'}\sqrt{\frac{\log 4/\delta}{n}} + \frac{1}{n}\left(2p'\log\frac{n}{\delta}\right)^{p'+1}\right)^{q} \log^{-1}\frac{4}{\delta} d\delta \\
&\leq (4p')^{p'q} + Cp'q\left(\frac{(16p')^{p'q}}{n^{\frac{q}{2}}}\Gamma(q/2) + \frac{(2p')^{(p'+1)q}}{n^q}\left((2\log n)^{(p'+1)q} + \Gamma\left((p'+1)q\right)\right)\right), \quad (46b)
\end{aligned}$$

where $\Gamma(\cdot)$ stands for the Gamma function. Combining the bounds (46a) and (46b), we reach to the following upper bound for $T_1(\theta, p')$:

$$\begin{aligned}
T_1(\theta,p') \leq \left(\frac{Cq}{n}\right)^{q/2}\Bigg[&(4p')^{p'q} + Cp'q\Bigg(\frac{(16p')^{pq}}{n^{\frac{q}{2}}}\Gamma(q/2) \\
&+ \frac{(2p')^{(p'+1)q}}{n^q}\left((2\log n)^{(p'+1)q} + \Gamma\left((p'+1)q\right)\right)\Bigg)\Bigg]. \quad (47)
\end{aligned}$$

Plugging the upper bounds of T_1 in equation (47) into equation (45) and taking the union bound over all $\theta_k \in \mathcal{N}\left(2^{-(p+2)}, \mathbb{S}^{d-1}, \|\cdot\|_2\right)$, we find that

$$\begin{aligned}
\mathbb{E}\left[\mathcal{R}^q(\mathbb{S}^{d-1})\right] &\leq 2^q\mathbb{E}\left[\mathcal{R}^q\left(\mathcal{N}(2^{-(p+2)})\right)\right] \\
&\leq 2^q p\left(2^{p+3}\right)^d \sup_{\theta\in\mathbb{S}^{d-1}, p'\in[1,p]} T_1(\theta,p') \\
&\leq 2^q p\left(2^{p+3}\right)^d\left(\frac{Cq}{n}\right)^{\frac{q}{2}}\Bigg[(4p)^{pq} + Cpq\Bigg(\frac{(16p)^{pq}}{n^{\frac{q}{2}}}\Gamma(q/2) \\
&\qquad + \frac{(2p)^{(p+1)q}}{n^q}\left((2\log n)^{(p+1)q} + \Gamma\left((p+1)q\right)\right)\Bigg)\Bigg],
\end{aligned}$$

for any given $u \in \mathcal{N}\left(\frac{1}{8}, \mathbb{S}^{d-1}, \|.\|_2\right)$. Taking the supremum over $u \in \mathcal{N}\left(\frac{1}{8}, \mathbb{S}^{d-1}, \|.\|_2\right)$ of both sides in the above bound and applying Minkowski's inequality, we obtain that

$$
\begin{aligned}
(\mathbb{E}|Z|^q)^{\frac{1}{q}} \leq & \left(\frac{64}{7}\right)^{d/q} \left(\mathbb{E}\left[\sup_{\theta \in \mathbb{S}^{d-1}} \left|\frac{2}{n}\sum_{i=1}^n \varepsilon_i Y_i X_i^\top u \left(X_i^\top \theta\right)^{p-1}\right|^q\right]\right)^{\frac{1}{q}} \\
\leq & 2\left(10 \cdot 2^{p+3}\right)^{d/q} \left[\sqrt{\frac{C_p q}{n}} + \frac{C_p q}{n} + \frac{C_p}{n^{\frac{3}{2}}} (\log n + q)^{p+1}\right],
\end{aligned}
$$

where C_p is a universal constant depending only on p. By choosing $q = d(p+7) + \log \frac{2}{\delta}$ and using Markov inequality, we find that

$$
\mathbb{P}\left(|Z| \geq C_p \left(\sqrt{\frac{d + \log \frac{1}{\delta}}{n}} + \frac{1}{n^{3/2}} \left(d + \log \frac{n}{\delta}\right)^{p+1}\right)\right) \leq \delta.
$$

Thus, we have establish the claim (42a).

Proof of claim (42b): In order to obtain a uniform concentration bound for J_2, we use an argument similar to that from the proof of claim (42a). In particular, since polynomial $(x^\top \theta)^{2p-1}$ is homogeneous in terms of θ, using the same normalization as in equation (43), it suffices to demonstrate that

$$
\mathbb{P}\left(W \leq c r^{2p-1} \left(\sqrt{\frac{d + \log \frac{1}{\delta}}{n}} + \frac{1}{n^{3/2}} \left(d + \log \frac{n}{\delta}\right)^{2p+1}\right)\right) \geq 1 - \delta, \tag{48}
$$

for any $\delta > 0$ where $W := \sup_{\theta \in \mathbb{S}^{d-1}} \left\|\frac{1}{n}\sum_{i=1}^n X_i \left(X_i^\top \theta\right)^{2p-1} - \mathbb{E}_X \left[X \left(X^\top \theta\right)^{2p-1}\right]\right\|_2$.

For each $u \in \mathbb{R}^d$, define the random variable

$$
W_u := \sup_{\theta \in \mathbb{S}^{d-1}} \left|\frac{1}{n}\sum_{i=1}^n X_i^\top u \left(X_i^\top \theta\right)^{2p-1} - \mathbb{E}_X \left[X^\top u \left(X^\top \theta\right)^{2p-1}\right]\right|
$$

It suffices to bound W_u for fixed $u \in \mathcal{N}\left(\frac{1}{8}, \mathbb{S}^{d-1}, \|.\|_2\right)$. We bound W_u by controlling its moments. By a symmetrization argument, we have

$$
\begin{aligned}
\mathbb{E}\left[\sup_{\theta \in \mathbb{S}^{d-1}} \left|\frac{1}{n}\sum_{i=1}^n X_i^\top u \left(X_i^\top \theta\right)^{2p-1} - \mathbb{E}_X \left[X^\top u \left(X^\top \theta\right)^{2p-1}\right]\right|^q\right] & \\
\leq \mathbb{E}\left[\sup_{\theta \in \mathbb{S}^{d-1}} \left|\frac{2}{n}\sum_{i=1}^n \varepsilon_i X_i^\top u \left(X_i^\top \theta\right)^{2p-1}\right|^q\right]. &
\end{aligned}
$$

From here, we can use the same technique as that in and after inequality (45) to bound the RHS term in the above display. Therefore, we will only highlight the main differences here. For any compact set $\Omega \subseteq \mathbb{R}^d$, we define the random variable

$$
\mathcal{Q}(\Omega) := \sup_{\theta \in \Omega, p' \in [1, p]} \left|\frac{2}{n}\sum_{i=1}^n \varepsilon_i X_i^\top u \left(X_i^\top \theta\right)^{2p'-1}\right|.
$$

Following the similar argument as that in equation (45), we can check that $\mathcal{Q}(\mathbb{S}^{d-1}) \leq 2\mathcal{Q}\left(\mathcal{N}(2^{-(2p+2)})\right)$. A direct application of union bound leads to

$$\mathbb{E}\left[\mathcal{Q}^q\left(\mathcal{N}(2^{-(2p+2)})\right)\right] \leq 2p \cdot \left|\mathcal{N}(2^{-(2p+2)})\right| \cdot \sup_{\theta\in\mathbb{S}^{d-1}, p'\in[1,p]} \underbrace{\mathbb{E}\left[\left(\left|\frac{4}{n}\sum_{i=1}^n \varepsilon_i X_i^\top u \left(X_i^\top \theta\right)^{2p'-1}\right|\right)^q\right]}_{:=T_2(\theta,p')}.$$

We control $T_2(\theta, p')$ using the same approach as that the proof of claim (42a). For the convenience of notation, we denote $h_{\theta,u}(x) := (x^\top u)^2 (x^\theta)^{2(2p'-1)}$. Simple algebra lead to the following upper bounds:

$$\mathbb{E}\left[h_{\theta,u}(X_i)\right] \leq (4p')^{2p'}, \qquad \mathbb{E}\left[h_{\theta,u}(X_i)^q\right] \leq (4p'q)^{2p'q}.$$

Invoking the result of Lemma 2, the above bounds lead to the following probability bound:

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n h_{\theta,u}(X_i) - \mathbb{E}_X\left[h_{\theta,u}(X)\right]\right| \leq (16p')^{2p'}\sqrt{\frac{\log 4/\delta}{n}} + \left(4p'\log\frac{n}{\delta}\right)^{2p'}\frac{\log 4/\delta}{n}\right) \leq \delta.$$

Therefore, we further obtain that

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n h_{\theta,u}(X_i)\right)^{q/2}\right] \leq (8p')^{2p'q} + Cp'q\Bigg(\frac{(32p')^{2p'q}}{n^{\frac{q}{2}}}\Gamma(q/2) + \frac{(4p')^{(2p'+1)q}}{n^q}\left((2\log n)^{(2p'+1)q} + \Gamma\left((2p'+1)q\right)\right)\Bigg).$$

Combining the above bound and an upper bound of $T_2(\theta, p')$ based on Khintchine's inequality, we obtain the following inequality:

$$T_2(\theta,p') \leq \left(\frac{Cq}{n}\right)^{q/2}\Bigg[(8p')^{2p'q} + Cp'q\Bigg(\frac{(32p')^{2p'q}}{n^{\frac{q}{2}}}\Gamma(q/2) + \frac{(4p')^{(2p'+1)q}}{n^q}\left((2\log n)^{(2p'+1)q} + \Gamma\left((2p'+1)q\right)\right)\Bigg)\Bigg].$$

Collecting the above bounds leads to

$$\begin{aligned}\mathbb{E}\left[\mathcal{Q}^q(\mathbb{S}^{d-1})\right] &\leq 2^{q+1}p\left(2^{2p+3}\right)^d \sup_{\theta\in\mathbb{S}^{d-1}, p'\in[1,p]} T_2(\theta,p') \\ &\leq 2^{q+1}p\left(2^{2p+3}\right)^d \left(\frac{Cq}{n}\right)^{\frac{q}{2}}\Bigg[(8p)^{2pq} + Cpq\Bigg(\frac{(32p)^{2pq}}{n^{\frac{q}{2}}}\Gamma(q/2) \\ &\qquad + \frac{(4p)^{(2p+1)q}}{n^q}\left((2\log n)^{(2p+1)q} + \Gamma\left((2p+1)q\right)\right)\Bigg)\Bigg],\end{aligned}$$

for any fixed $u \in \mathcal{N}\left(\frac{1}{8}, \mathbb{S}^{d-1}, \|.\|_2\right)$. Taking supremum over $u \in \mathcal{N}\left(\frac{1}{8}, \mathbb{S}^{d-1}, \|.\|_2\right)$ of both sides in the above bound and applying Minkowski's inequality, we arrive at the following bound:

$$\begin{aligned}\left(\mathbb{E}\left[|W|^q\right]\right)^{\frac{1}{q}} &\leq \left(\frac{64}{7}\right)^{\frac{d}{q}}\left(\mathbb{E}\left[\sup_{\theta\in\mathbb{S}^{d-1}}\left|\frac{2}{n}\sum_{i=1}^n \sigma_i X_i^\top u\left(X_i^\top\theta\right)^{2p-1}\right|^q\right]\right)^{\frac{1}{q}} \\ &\leq \left(\frac{10}{\varepsilon}\right)^{\frac{d}{q}}\left[\sqrt{\frac{C_p q}{n}} + \frac{C_p q}{n} + \frac{C_p}{n^{\frac{3}{2}}}(\log n + q)^{2p+1}\right],\end{aligned}$$

where C_p is a universal constant depending only upon p. With the choice of $q = d(2p+7)+\log\frac{2}{\delta}$, we obtain that

$$\mathbb{P}\left(|W| \geq C_p\left(\sqrt{\frac{d+\log\frac{1}{\delta}}{n}} + \frac{1}{n^{3/2}}\left(d+\log\frac{n}{\delta}\right)^{2p+1}\right)\right) \leq \delta.$$

Thus, we have established the claim (42b).

A.3 Proof of Corollary 4

The proof of Corollary 4 follows directly by verifying the claims (29a) and (29b).

A.3.1 Structure of F^G

Direct algebra leads to the following equation

$$\begin{aligned}\langle \nabla F^G(\theta),\, \theta^* - \theta\rangle &= \left(\theta - \mathbb{E}\left[X\tanh\left(X^\top\theta\right)\right]\right)^\top(\theta - \theta^*) \\ &\geq \|\theta\|_2^2 - \|\theta\|_2\left\|\mathbb{E}\left[X\tanh\left(X^\top\theta\right)\right]\right\|_2 \end{aligned} \tag{49}$$

where $\tanh(x) := \frac{\exp(x)-\exp(-x)}{\exp(x)+\exp(-x)}$ for all $x \in \mathbb{R}$. From Theorem 2 in Dwivedi et al. [9], we have

$$\left\|\mathbb{E}\left[X\tanh\left(X^\top\theta\right)\right]\right\|_2 \leq \left(1 - p + \frac{p}{1+\frac{\|\theta\|_2^2}{2}}\right)\|\theta\|_2$$

for all $\theta \in \mathbb{R}^d$ where $p := \mathbb{P}\left(|Y| \leq 1\right) + \frac{1}{2}\mathbb{P}\left(|Y| > 1\right)$ where $Y \sim \mathcal{N}(0,1)$. Plugging the above inequality into equation (49) leads to

$$\langle \nabla F^G(\theta),\, \theta^* - \theta\rangle \geq \frac{p\,\|\theta\|_2^4}{2+\|\theta\|_2^2} \geq \begin{cases} \frac{p}{4}\,\|\theta\|_2^4, & \text{for } \|\theta\|_2 \leq \sqrt{2} \\ \frac{p}{2}\left(\|\theta\|_2^2 - 1\right), & \text{otherwise}\end{cases}.$$

As a consequence, we achieve the conclusion of claim (29a).

A.3.2 Perturbation error between ∇F^G and ∇F_n^G

Direct calculation indicates the following equation:

$$\nabla F_n^G(\theta) - \nabla F^G(\theta) = \frac{1}{n}\sum_{i=1}^n X_i\tanh(X_i^\top\theta) - \mathbb{E}\left[X\tanh\left(X^\top\theta\right)\right].$$

The outer expectation in the above display is taken with respect to $X \sim \mathcal{N}(\theta^*, \sigma^2 I_d)$ where $\theta^* = 0$. Based on the proof argument of Lemma 1 from the paper [9], for each $r > 0$, we have the following concentration inequality

$$\mathbb{P}\left(\sup_{\theta\in\mathbb{B}(\theta^*,r)}\left\|\frac{1}{n}\sum_{i=1}^n X_i\tanh(X_i^\top\theta) - \mathbb{E}\left[X\tanh\left(X^\top\theta\right)\right]\right\|_2 \leq cr\sqrt{\frac{d+\log(1/\delta)}{n}}\right) \geq 1-\delta, \tag{50}$$

for any $\delta > 0$ as long as the sample size $n \geq c' d \log(1/\delta)$ where c and c' are universal constants. For any $M \in \mathbb{N}_+$, by the concentration bound (50) and the union bound, we find that

$$\mathbb{P}\left(\forall r \in [2^{-M}, 1], \sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \nabla F_n^G(\theta) - F^G(\theta) \right\|_2 \leq c\, r \sqrt{\frac{d + \log(M/\delta)}{n}} \right) \geq 1 - \delta. \tag{51}$$

On the other hand, based on the standard inequality $|\tanh(x)| \leq |x|$ for all $x \in \mathbb{R}$, we find that

$$\begin{aligned} \left\| \nabla F_n^G(\theta) - \nabla F^G(\theta) \right\|_2 &\leq \frac{1}{n} \sum_{i=1}^n \|X_i\|_2 \left| \tanh\left(X_i^\top \theta \right) \right| + \mathbb{E}\left[\|X\|_2 \left| \tanh\left(X^\top \theta \right) \right| \right] \\ &\leq \frac{1}{n} \sum_{i=1}^n \|X_i\|_2 \left| X_i^\top \theta \right| + \mathbb{E}\left[\|X\|_2 \left| X^\top \theta \right| \right] \\ &\leq \left(\frac{1}{n} \sum_{i=1}^n \|X_i\|_2^2 + \mathbb{E}\left[\|X\|_2^2 \right] \right) \|\theta\|_2 . \end{aligned}$$

Therefore, we have $\left\| \nabla F_n^G(\theta) - \nabla F^G(\theta) \right\|_2 \leq 2d \|\theta\|_2 \log(1/\delta)$ with probability $1 - \delta$. By choosing $M_1 := \log(2nd)$, based on the previous bound, we obtain that

$$\mathbb{P}\left(\forall r < 2^{-M_1}, \sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \nabla F_n^G(\theta) - \nabla F^G(\theta) \right\|_2 \leq \frac{\log(1/\delta)}{n} \right) \geq 1 - \delta. \tag{52}$$

Furthermore, for vector $\theta \in \mathbb{R}^d$ with large norm, by the concentration bound (50) combined with the union bound, for any $M' \in \mathbb{N}_+$, we find that

$$\mathbb{P}\left(\forall r \in [1, 2^{M'}], \sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \nabla F_n^G(\theta) - F^G(\theta) \right\|_2 \leq c\, r \sqrt{\frac{d + \log(M'/\delta)}{n}} \right) \geq 1 - \delta.$$

When r in the above bound is too large, we can simply use the fact that tanh is a bounded function. We thus have the upper bound

$$\left\| \nabla F_n^G(\theta) - \nabla F^G(\theta)) \right\|_2 \leq \mathbb{E}\left[\|X\|_2 \right] + \frac{1}{n} \sum_{i=1}^n \|X_i\|_2 ,$$

for any θ. Given the above bound, by choosing $M_2 := \log(2\sqrt{n})$, we obtain that

$$\begin{aligned} &\mathbb{P}\left(\forall r > 2^{M_2}, \sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \nabla F_n^G(\theta) - \nabla F^G(\theta)) \right\|_2 \leq r \sqrt{\frac{d + \log(1/\delta)}{n}} \right) \\ &\qquad \geq \mathbb{P}\left(\mathbb{E}\left[\|X\|_2 \right] + \frac{1}{n} \sum_{i=1}^n \|X_i\|_2 \leq 2^{M_2} \sqrt{\frac{d + \log(1/\delta)}{n}} \right) \geq 1 - \delta. \end{aligned} \tag{53}$$

Putting the bounds (51), (52), and (53) together, for $n \geq cd \log(1/\delta)$, the following probability bound holds

$$\mathbb{P}\left(\forall r > 0, \sup_{\theta \in \mathbb{B}(\theta^*, r)} \left\| \nabla F_n^G(\theta) - \nabla F^G(\theta)) \right\|_2 \leq c\, r \sqrt{\frac{d + \log\left(\log n/\delta\right)}{n}} + \frac{\log(1/\delta)}{n} \right) \geq 1 - \delta,$$

which completes the proof of the claim (29b).

B Proofs of some auxiliary results

In this appendix, we state and prove a few technical lemmas used in the proofs of our main results.

B.1 A limit result

We begin with a lemma in the limiting behavior of a function. The lemma is used in the proof of Theorem 2 in Section 5.2.

Lemma 1. *Let ϕ be a non-increasing continuous function on the real line with $\phi(c) = 0$, and such that $\phi(t) \geq 0$ for all $t \in (c, \infty)$. Suppose that there exist two continuous functions $f, g : [0, +\infty) \to \mathbb{R}$ such that $\lim_{t\to+\infty} g(t)$ exists and $f(t) \leq \int_0^t \phi(g(s))ds$ for all $t \geq 0$. Under these conditions, we have $\lim_{t\to+\infty} g(t) \leq c$.*

Proof. Define the limit $A := \lim_{t\to+\infty} g(t)$, which exists according to the assumptions. We proceed via proof by contradiction. In particular, suppose that $A > c$. Based on the definition of A, for the positive constant $\varepsilon = (A - c)/2 > 0$, we can find a sufficiently large positive constant T such that $g(t) > A - \varepsilon$ for any $t \geq T$. According to the assumptions on ϕ, we obtain that

$$\delta := - \sup_{s \geq c+\varepsilon} \phi(s) < 0.$$

Therefore, for all $t > T$, we arrive at the following inequalities

$$0 \leq f(t) \leq \int_0^T \phi(g(s))ds + \int_T^t \phi(g(s))ds \leq \int_0^T \phi(g(s))ds - \delta(t - T).$$

By choosing $t = 1 + T + \delta^{-1} \int_0^T \phi(g(s))ds$, the above inequality cannot hold. This yields the desired contradiction, which completes the proof. □

B.2 A tail bound based on truncation

We now state an upper deviation inequality based on a truncation argument. This lemma is used in Appendix A.2 to prove the uniform concentration bound (24b). Consider a sequence of random variables $\{Y_i\}_{i=1}^n$ satisfying the moment bounds

$$\mathbb{E}\left[|Y_i|^q\right] \leq (aq)^{bq} \quad \text{for all } q = 1, 2, \ldots \tag{54}$$

where a, b are universal constants.

Lemma 2. *Given an i.i.d. sequence of zero-mean random variables $\{Y_i\}_{i=1}^n$ satisfying the moment bounds (54), we have*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n Y_i \geq (4a)^b \sqrt{\frac{\log 4/\delta}{n}} + \left(a \log \frac{n}{\delta}\right)^b \frac{\log 4/\delta}{n}\right) \leq \delta.$$

Proof. The proof of the lemma is a direct combination of truncation argument and Bernstein's inequality. In particular, for each $i \in [n]$, define the truncated random variable $\tilde{Y}_i := Y_i \mathbb{I}\left[|Y_i| \leq 3(a \log \frac{n}{\delta})^b\right]$. With this definition, we have

$$\mathbb{P}\left((Y_i)_{i=1}^n \neq (\tilde{Y}_i)_{i=1}^n\right) = \mathbb{P}\left(\max_{1\leq i\leq n} |Y_i| > 3\left(a \log \frac{n}{\delta}\right)^b\right) \leq n\mathbb{P}\left(|Y_i| > 3\left(a \log \frac{n}{\delta}\right)^b\right) \leq \frac{\delta}{2}.$$

Therefore, it is sufficient to study a concentration behavior of the quantity $\sum_{i=1}^n \tilde{Y}_i$. Invoking Bernstein's inequality [4], we obtain that

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n \tilde{Y}_i \geq \varepsilon\right) \leq 2\exp\left(-\frac{n\varepsilon^2}{2(2a)^{2b} + \frac{2}{3}\varepsilon \cdot 3(a\log\frac{n}{\delta})^b}\right).$$

In order to make the RHS of the above inequality less than $\frac{\delta}{2}$, it suffices to set

$$\varepsilon = (4a)^b\sqrt{\frac{\log(4/\delta)}{n}} + \left(a\log\frac{n}{\delta}\right)^b \frac{\log(4/\delta)}{n}.$$

Collecting all of the above inequalities yields the claim. □

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